RANK-LEVEL DUALITY OF CONFORMAL BLOCKS FOR ODD ORTHOGONAL LIE ALGEBRAS IN GENUS 0.

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ABSTRACT. Classical invariants for representations of one Lie group can often be related to invariants of some other Lie group. Physics suggests that the right objects to consider for these questions are certain refinements of classical invariants known as conformal blocks. Conformal blocks appear in algebraic geometry as spaces of global sections of line bundles on the moduli stack of parabolic bundles on a smooth curve. Rank-level duality connects a conformal block associated to one Lie algebra to a conformal block for a different Lie algebra. In this paper we prove a rank-level duality for type $\mathfrak{so}(2r+1)$ on the pointed projective line conjectured by T. Nakanishi and A. Tsuchiya.

1. INTRODUCTION

It has been known for a long time that invariant theory of GL_r and the Intersection theory of Grassmanians are related. This relation gives rise to some interesting isomorphisms between invariants of SL_r and SL_s for some positive integer s. To make it precise recall that the irreducible polynomial representations of GL_r are indexed by r-tuple of integers $\lambda = (\lambda^1 \geq \cdots \geq \lambda^r \geq 0) \in \mathbb{Z}^r$. Let V_λ denote the corresponding irreducible GL_r module.

Consider $\lambda = (\lambda^1 \geq \cdots \geq \lambda^r \geq 0)$ an r-tuple of integers such that $\lambda^1 \leq s$. The set of all such λ's are in bijections with $\mathcal{Y}_{r,s}$, the set of all Young diagrams with at most r rows and s columns. For λ, μ, ν in $\mathcal{Y}_{r,s}$ such that $|\lambda| + |\mu| + |\nu| = rs$ we know that

$$\dim_{\mathbb{C}}(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{SL}_r} = \dim_{\mathbb{C}}(V_{\lambda^T} \otimes V_{\mu^T} \otimes V_{\nu^T})^{\mathrm{SL}_s},$$

where $|\lambda|$ denote the number of boxes in the Young diagram of λ and λ^T denotes the transpose of the Young diagram of λ. The above is not only a numerical "strange" duality but the vector spaces are canonically dual to each other (see [5]).

Physics suggests that to understand the above kind of relations for other groups the correct objects to consider are certain refinements of the co-invariants known as a conformal blocks. Consider a finite dimensional simple complex Lie algebra $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$ and a non-negative integer ℓ called the level. Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be an n-tuple of dominant weight of $\mathfrak{g}$ of level ℓ. To n distinct points $\vec{p} = (P_1, \ldots, P_n)$ with coordinates $\vec{z} = (z_1, \ldots, z_n)$ on $\mathbb{P}^1$ one associates a finite dimensional vector space $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{g}, \ell, \vec{z})$ known as a conformal block. More generally, one can define conformal blocks associated to the n-distinct points on curves with at most nodal singularities (see Section 2) of arbitrary genus. Conformal blocks organize themselves to form a vector bundle on $\overline{\mathrm{M}}_{g,n}$, the moduli stack of stable n-pointed curves of genus g.

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Rank-level duality is a duality between conformal blocks associated to two different Lie algebras. In [16], T. Nakanishi and A. Tsuchiya proved that on $\mathbb{P}^1$, certain conformal blocks of $\mathfrak{sl}(r)$ at level s are dual to conformal blocks of $\mathfrak{sl}(s)$ at level r. In [1], T. Abe proved a rank-level duality statements between conformal blocks of type $\mathfrak{sp}(2r)$ at level s and $\mathfrak{sp}(2s)$ at level r. It is important to point out that there is no known relation between the classical invariants for the Lie algebras $\mathfrak{sp}(2r)$ and $\mathfrak{sp}(2s)$.

The paper [16] suggests that one can try to answer similar rank-level duality questions for orthogonal Lie algebras on $\mathbb{P}^1$. Further, it is also pointed out in [16] that one should only consider the tensor representations i.e. representations that lift to representations of the special orthogonal group (see Section 6 in [16]). In the following we answer the above question for odd orthogonal Lie algebras.

Throughout this paper we assume that $r, s \geq 3$. Let $P^0_{2s+1}(\mathfrak{so}(2r+1))$ denote the set of tensor representations of $\mathfrak{so}(2r+1)$ of level $2s+1$. We can realize the set $P^0_{2s+1}(\mathfrak{so}(2r+1))$ as a disjoint union of $\mathcal{Y}_{r,s}$ and $\sigma(\mathcal{Y}_{r,s})$, where σ is an involution $P^0_{2s+1}(\mathfrak{so}(2r+1))$ that corresponds to action of the diagram automorphism. Our main theorem is the following:

Theorem 1.1. *Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathcal{Y}^n_{r,s}$ be a n-tuple of weights in $P^0_{2s+1}(\mathfrak{so}(2r+1))$, then the following conformal blocks are isomorphic.*

(1) *If $\sum_{i=1}^n |\lambda_i|$ is even, then*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{so}(2r+1), 2s+1, \vec{z}) \simeq \mathcal{V}^{\dagger}_{\vec{\lambda}^T}(\mathfrak{so}(2s+1), 2r+1, \vec{z}),$$

where $\vec{z}$ denotes n-distinct points on $\mathbb{P}^1$.

(2) *If $\sum_{i=1}^n |\lambda_i|$ is odd, then*

$$\mathcal{V}_{\vec{\lambda},0}(\mathfrak{so}(2r+1), 2s+1, \vec{z}) \simeq \mathcal{V}^{\dagger}_{\vec{\lambda}^T,\sigma(0)}(\mathfrak{so}(2s+1), 2r+1, \vec{z}),$$

where $\vec{z}$ denotes $n+1$-distinct points on $\mathbb{P}^1$.

(3) *If $\sum_{i=1}^n |\lambda_i|$ is even, then*

$$\mathcal{V}_{\vec{\lambda},\sigma(0)}(\mathfrak{so}(2r+1), 2s+1, \vec{z}) \simeq \mathcal{V}^{\dagger}_{\vec{\lambda}^T,\sigma(0)}(\mathfrak{so}(2s+1), 2r+1, \vec{z}),$$

where $\vec{z}$ denotes $n+1$-distinct points on $\mathbb{P}^1$.

Remark 1.2. *The above statements are independent of each other.*

We briefly discuss the general context of rank-level duality maps. We closely follow the methods used in the rank-level duality of conformal blocks in [16], [1] and [3] but there are significant differences in key steps.

Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}$ are simple Lie algebras and consider an embedding of Lie algebras $\phi : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}$. We extend it to a map of affine Lie algebras $\widehat{\phi} : \widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2 \to \widehat{\mathfrak{g}}$. Consider a level 1 integrable highest weight module $\mathcal{H}_\Lambda(\mathfrak{g})$ and restrict it to $\widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2$. The module $\mathcal{H}_\Lambda(\mathfrak{g})$ decomposes into irreducible integrable $\widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2$ modules of level $\ell = (\ell_1, \ell_2)$

$$\bigoplus_{(\lambda,\mu)\in B(\Lambda)} m^\Lambda_{\lambda,\mu} \mathcal{H}_\lambda(\mathfrak{g}_1) \otimes \mathcal{H}_\mu(\mathfrak{g}_2) \simeq \mathcal{H}_\Lambda(\mathfrak{g}),$$

where ℓ is the Dynkin multi index of ϕ and $m^\Lambda_{\lambda,\mu}$ is the multiplicity of the component $\mathcal{H}_\lambda(\mathfrak{g}_1) \otimes \mathcal{H}_\mu(\mathfrak{g}_2)$. In general the number of components $B(\Lambda)$ may be infinite. We only consider those

embeddings such that $|B(\Lambda)|$ is finite. These embeddings are known as conformal embeddings (see [13] for more details).

Further assume that $m^{\Lambda}_{\lambda,\mu}=1$ for all level 1 weight Λ. Thus for a n-tuple $\vec{\Lambda}=(\Lambda_1,\dots,\Lambda_n)$ of level 1 dominant weights of $\mathfrak{g}$ we get a map

$$\bigotimes_{i=1}^{n}(\mathcal{H}_{\lambda_i}(\mathfrak{g}_1)\otimes\mathcal{H}_{\mu_i}(\mathfrak{g}_2))\rightarrow\bigotimes_{i=1}^{n}\mathcal{H}_{\Lambda_i}(\mathfrak{g}).$$

We take n-distinct points $\vec{z}$ on $\mathbb{P}^1$ and taking "coinvariants" we get a map

$$\alpha:\mathcal{V}_{\vec{\lambda}}(\mathfrak{g}_1,\ell_1,\vec{z})\otimes\mathcal{V}_{\vec{\mu}}(\mathfrak{g}_2,\ell_2,\vec{z})\rightarrow\mathcal{V}_{\vec{\Lambda}}(\mathfrak{g},1,\vec{z}),$$

where $\vec{\lambda}=(\lambda_1,\dots,\lambda_n)$ and $\vec{\mu}=(\mu_1,\dots,\mu_n)$. If $\dim_{\mathbb{C}}(\mathcal{V}_{\vec{\Lambda}}(\mathfrak{g},1,\vec{z}))=1$, we get a map $\mathcal{V}_{\vec{\lambda}}(\mathfrak{g}_1,\ell_1,\vec{z})\rightarrow\mathcal{V}^{\dagger}_{\vec{\mu}}(\mathfrak{g}_2,\ell_2,\vec{z})$. This map is known as the rank-level duality map. The above analysis with the embedding $\widehat{\mathfrak{so}}(2r+1)\oplus\widehat{\mathfrak{so}}(2s+1)\rightarrow\widehat{\mathfrak{so}}((2r+1)(2s+1))$ gives the maps considered in Theorem 1.1. In Section 3, we define the rank-level duality for conformal blocks on n-pointed of nodal curves of arbitrary genus.

Remark 1.3. *Though the conformal blocks in Theorem 1.1 can be identified with the space of global sections of a line bundle on the moduli stack of* Spin*-bundles over $\mathbb{P}^1$ with parabolic structures on marked points, I was not able to define the rank-level duality map in Theorem 1.1 geometrically.*

We now discuss the main body of the proof of Theorem 1.1. This can be broken up into several steps:

1.0.1. *Dimension Check.* The Verlinde formula gives us the dimension of the conformal block. Using the Verlinde formula we show that the dimensions of the source and the target of the conformal blocks in Theorem 1.1 are the same. Unlike the case in [1] we do not have a bijection between $P_{2s+1}(\mathfrak{so}(2r+1))$ and $P_{2r+1}(\mathfrak{so}(2s+1))$ but we get around the problem by considering bijection of the orbits of $P_{2s+1}(\mathfrak{so}(2r+1))$ and $P_{2r+1}(\mathfrak{so}(2s+1))$ under the involution σ as described in [17]. If $\vec{\lambda}\in\mathcal{Y}^n_{r,s}$ and $\Gamma=\{1,\sigma\}$ be the group acting on $P_{2s+1}(\mathfrak{so}(2r+1))$, the Verlinde formula in this case takes the form

$$\sum_{\mu\in P_{2s+1}(\mathfrak{so}(2r+1))/\Gamma}|\operatorname{Orb}_{\mu}|f(\mu,\vec{\lambda})$$

where $f(\mu,\vec{\lambda})$ is a function, constant on the orbits and $|\operatorname{Orb}_{\mu}|$ denotes the cardinality of the orbit of μ under the action of Γ. Using a non trivial trigonometric identity in [17] and a generalization of Lemma A.42 in [10], we show that the orbit sum $|\operatorname{Orb}_{\mu}|f(\mu,\vec{\lambda})$ is same for the corresponding orbit sum for the Lie algebra $\mathfrak{so}(2s+1)$ at level $2r+1$.

1.0.2. *Flatness of rank-level duality.* The rank-level duality map has constant rank when the points $\vec{z}$ varies (see [6]). The conformal embedding is important in this case as it ensures that the rank-level duality map is flat with respect to the $KZ/Hitchin/WZW$ connection (see [6]) on sheaves of vacua over any family of smooth curves.

1.0.3. *Degeneration of a smooth family.* Let $C_1 \cup C_2$ be a nodal curve where C_1 and C_2 are isomorphic to $\mathbb{P}^1$ intersecting at one point. The conformal block on $C_1 \cup C_2$ is isomorphic to a direct sum of conformal blocks on the normalization of $C_1 \cup C_2$. This property is known as factorization of conformal blocks. A key ingredient of the proof of rank-level duality in [1] is the compatibility of the rank-level duality with factorization. T. Abe uses it to conclude that the rank-level duality map is an isomorphism on certain nodal curves.

This property for nodal curves is no longer true for our present case due to the presence of "non-classical" components(i.e. components that do not appear in the branching of the finite dimensional irreducible module) in the branching of highest weight integrable modules. We refer the reader to Section 3 for more details.

We consider a family of smooth curves degenerating to a nodal curve X_0. Instead of looking at the nature of the rank-level duality map on the nodal curve we study the nature of the rank-level duality map on nearby smooth curves of the nodal curve X_0 under any conformal embedding. We use the "sewing procedure" of [21] to understand the decomposition of the rank-level duality map near the nodal curve X_0. The methods used in this step are similar to [3]. This degeneration technique and the flatness of the rank-level duality enables us to use induction similar to [16] and [1] to reduce to the case for one dimensional conformal blocks on $\mathbb{P}^1$ with three marked points.

1.0.4. *Minimal Cases.* We are now reduced to show that the rank-level duality maps for one dimensional conformal blocks on $\mathbb{P}^1$ with three marked points are non-zero. Our proof of this step differs significantly from that in [1] as we were not able to use any geometry of parabolic vector bundles with a non-degenerate form. This is again due to the presence of non-classical components. Using [13], we construct explicit vectors $v_1 \otimes v_2 \otimes v_3$ in the tensor product of three highest weights module and show by using the "Gauge Symmetry" that $\Psi(v_1 \otimes v_2 \otimes v_3) \neq 0$. It will be very interesting if one can define that rank-level duality map in this case purely in language of vector bundles with a non degenerate form.

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2. Basic Definitions in the theory of Conformal blocks

We recall some basic definitions from [21] in the theory of conformal blocks. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We fix the decomposition of $\mathfrak{g}$ into root spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where Δ is the set of roots decomposed into a union of $\Delta_+ \cup \Delta_-$ of positive and negative roots. Let $(,)$ denote the Cartan Killing form on $\mathfrak{g}$ normalized such that $(\theta, \theta) = 2$ where θ is the longest root and we identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using the form $(,)$.

2.1. **Affine Lie algebras.** We define the affine Lie algebra $\widehat{\mathfrak{g}}$ to be

$$\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C}c$$

where c belong to the center of $\widehat{\mathfrak{g}}$ and the Lie bracket is given as follows:

$$[X \otimes f(\xi), Y \otimes g(\xi)] = [X, Y] \otimes f(\xi)g(\xi) + (X, Y) \operatorname{Res}_{\xi=0}(gdf).c,$$

where $X, Y \in \mathfrak{g}$ and $f(\xi), g(\xi) \in \mathbb{C}((\xi))$.

Let $X(n) = X \otimes \xi^n$ and $X = X(0)$ for any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$. The finite dimensional Lie algebra $\mathfrak{g}$ can be realized as a subalgebra of $\widehat{\mathfrak{g}}$ under the identification of X with $X(0)$.

2.2. **Representation Theory of affine Lie algebras.** The finite dimensional irreducible modules of $\mathfrak{g}$ are parametrized by the set of dominant integral weights $P_+ \subset \mathfrak{h}^*$. Let V_λ denote the irreducible module of highest weight $\lambda \in P_+$ and v_λ denote the highest weight vector.

We fix a positive integer ℓ which we call the level. The set of dominant integral weights of level ℓ is defined as follows

$$P_\ell(\mathfrak{g}) := \{\lambda \in P_+ | (\lambda, \theta) \leq \ell)\}$$

For each $\lambda \in P_\ell(\mathfrak{g})$ there is a unique irreducible integrable highest weight $\widehat{\mathfrak{g}}$ module $\mathcal{H}_\lambda$ which satisfies the following properties:

(1) $V_\lambda \subset \mathcal{H}_\lambda(\mathfrak{g})$
(2) The central element c of $\widehat{\mathfrak{g}}$ acts by the scalar ℓ.
(3) Let v_λ denote a highest weight vector in V_λ then

$$X_\theta(-1)^{\ell-(\theta,\lambda)+1} v_\lambda = 0,$$

where X_θ is a non-zero element in the weights space of $\mathfrak{g}_\theta$. More over $\mathcal{H}_\lambda(\mathfrak{g})$ is generated by V_λ over $\widehat{\mathfrak{g}}$ with the above relation. When $\lambda = 0$, the corresponding $\widehat{\mathfrak{g}}$-module $\mathcal{H}_0(\mathfrak{g})$ is known as Vaccum representation.

2.3. **Conformal Blocks.** We fix a n-pointed curve C with formal neighborhood $\eta_1, \dots, \eta_n$ around the n-points $\vec{p} = (P_1, \dots, P_n)$ which satisfies the following properties

(1) The curve C has at most nodal singularities.
(2) The curve C is smooth at the point $P_1, \dots, P_n$.
(3) $C - P_1, \dots, P_n$ is an affine curve.
(4) A stability condition (equivalent to the finiteness of the automorphisms of the pointed curve)
(5) Isomorphism $\eta_i : \widehat{\mathcal{O}}_{C,P_i} \simeq \mathbb{C}[[\xi]]$ for $i = 1, \dots, n$.

We denote by $\mathfrak{X} = (C; \vec{p}; \eta_1 \dots \eta_n)$ the above data associated to the curve C. We define another Lie algebra

$$\widehat{\mathfrak{g}}_n := \bigoplus_{i=1}^{n} \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\xi_i)) \oplus \mathbb{C}c$$

where c belongs to the center of $\widehat{\mathfrak{g}}_n$ and the Lie bracket is given as follows:

$$[\sum_{i=1}^{n} X_i \otimes f_i, \sum_{i=1}^{n} Y_i \otimes g_i] := \sum_{i=1}^{n} [X_i, Y_i] \otimes f_i g_i + \sum_{i=1}^{n} (X_i, Y_i) \operatorname{Res}_{\xi_i=0}(g_i df_i).c$$

We define the block algebra to be $\mathfrak{g}(\mathfrak{X}) := \mathfrak{g} \otimes \Gamma(C - \{P_1, \ldots, P_n\}, \mathcal{O}_C)$. By local expansion of functions using the chosen coordinates ξ_i we get the following embedding

$$\mathfrak{g}(\mathfrak{X}) \hookrightarrow \widehat{\mathfrak{g}}_n$$

Consider n-tuple of weights $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in P^n_\ell(\mathfrak{g})$. We set $\mathcal{H}_{\vec{\lambda}}(\mathfrak{g}) = \mathcal{H}_{\lambda_1}(\mathfrak{g}) \otimes \cdots \otimes \mathcal{H}_{\lambda_n}(\mathfrak{g})$. The algebra $\widehat{\mathfrak{g}}_n$ acts on $\mathcal{H}_{\vec{\lambda}}(\mathfrak{g})$. For any $X \in \mathfrak{g}$ and $f \in \mathbb{C}((\xi_i))$, the endomorphism $\rho_i(X \otimes f(\xi_i))$ is given by the following:

$$\rho_i(X \otimes f(\xi_i))|v_1\rangle \otimes \cdots \otimes |v_n\rangle = |v_1\rangle \otimes X \otimes f((\xi_i)|v_i\rangle \otimes |v_n\rangle$$

where $|v_i\rangle \in \mathcal{H}_{\lambda_i}$ for each i.

Definition 2.1. *We define the space of conformal blocks*

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}) := \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}/\mathfrak{g}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}, \mathbb{C})$$

We define the space of dual conformal blocks, $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}) = \mathcal{H}_{\vec{\lambda}}/\mathfrak{g}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}$. These are both finite dimensional $\mathbb{C}$-vector spaces which can be defined in families. The dimensions of these vector spaces are given by the Verlinde Formula.

We will follow Dirac's bra-ket notations. The elements of $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g})$(or $\mathcal{H}^*_{\vec{\lambda}}$) will be denoted by $\langle\Psi|$ and those of the dual conformal blocks (or $\mathcal{H}_{\vec{\lambda}}$) by $|\Phi\rangle$ and the pairing by $\langle\Psi|\Phi\rangle$.

Remark 2.2. *Let $X \in \mathfrak{g}$ and $f \in \Gamma(C - \{P_1, \ldots, P_n\}, \mathcal{O})$, then every element of $\langle\Psi| \in \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g})$ satisfies the following "Gauge Conditions":*

$$\sum_{i=1}^{n} \langle\Psi|\rho_i(X \otimes f(\xi_i))\Phi\rangle = 0,$$

2.4. **Propagation of Vacua.** Let P_{n+1} be a new point on the curve C with coordinate η_{n+1} and $\mathfrak{X}'$ denote the data. We associate the vaccum representation $\mathcal{H}_0$ to the point P_{n+1} and $\vec{\lambda}' = \vec{\lambda} \cup \{\lambda_{n+1} = 0\}$. The "propagation of vaccum" gives an isomorphism

$$f : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}', \mathfrak{g}) \to \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g})$$

where

$$f(\langle\Psi'|)|\Phi\rangle := \langle\Psi'|\Phi \otimes 0\rangle,$$

where $|0\rangle$ is a highest weight vector of the representation $\mathcal{H}_0$, $|\phi\rangle \in \mathcal{H}_{\vec{\lambda}}$ and $\langle\Psi'|$ is an arbitrary element of $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}', \mathfrak{g})$

2.5. **Conformal Blocks in a Family.** Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\vec{\lambda} \in P^n_\ell(\mathfrak{g})$. Consider a family $\mathcal{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \ldots; s_n, \xi_1, \ldots, \xi_n)$ of nodal curves on a base $\mathcal{B}$ with sections s_i and formal coordinates ξ_i. In [21], a locally free sheaf $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F}, \mathfrak{g})$ known as sheaf of Conformal blocks is constructed over the base $\mathcal{B}$. It is constructed as a subsheaf of $\mathcal{O}_{\mathcal{B}} \otimes \mathcal{H}^\dagger_{\vec{\lambda}}$. The sheaf $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F}, \mathfrak{g})$ commutes with base change. Similarly one can define another locally free sheaf $\mathcal{V}_{\vec{\lambda}}(\mathcal{F}, \mathfrak{g})$ as a quotient of $\mathcal{O}_{\mathcal{B}} \otimes \mathcal{H}_{\vec{\lambda}}$

More over if $\mathcal{F}$ is a smooth family of projective curves, then the sheaf $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F})$ carries a projectively flat connection known as the $KZ/Hitchin/WZW$ connection. We refer the reader to [21] for more details.

Remark 2.3. *When the level ℓ becomes unclear we also include it in the notation of conformal blocks. So if $\mathfrak{X}$ is the data associated to a n-pointed curve with chosen coordinates and we consider n-tuple of level ℓ weights $\vec{\lambda}$ of the Lie algebra $\mathfrak{g}$, we denote the conformal block by $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X},\mathfrak{g},\ell)$ and the dual conformal block by $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X},\mathfrak{g},\ell)$.*

3. Conformal Subalgebras and Rank-level duality map

In this section we discuss conformal embeddings of Lie algebras and give a general formulation of the rank-level maps.

3.1. **Conformal Embedding.** Let $\mathfrak{s}$, $\mathfrak{g}$ be two simple Lie algebra and $\phi:\mathfrak{s}\rightarrow\mathfrak{g}$ be an embedding of Lie algebras. Let $(,)_{\mathfrak{s}}$ and $(,)_{\mathfrak{g}}$ denote the normalized Cartan killing form such that the the length of the highest root is 2. We define the Dynkin-index of ϕ to be the unique integer d_{ϕ} such satisfying

$$(\phi(x),\phi(y))_{\mathfrak{g}}=d_{\phi}(x,y)_{\mathfrak{s}}$$

for all $x,y\in\mathfrak{s}$. When $\mathfrak{s}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ is semisimple, we define the Dynkin-multi index of $\phi=\phi_1\oplus\phi_2:\mathfrak{g}_1\oplus\mathfrak{g}_2\rightarrow\mathfrak{g}$ to be $d_{\phi}=(d_{\phi_1},d_{\phi_2})$.

If $\mathfrak{g}$ is simple, we define for any level ℓ and a dominant weight λ of level ℓ, the conformal anomaly $c(\mathfrak{g},\ell)$ and the trace anomaly $\Delta_{\lambda}(\mathfrak{g})$ as

$$c(\mathfrak{g},\ell)=\frac{\ell\dim\mathfrak{g}}{g^*+\ell}\text{ and }\Delta_{\lambda}=\frac{(\lambda,\lambda+2\rho)}{2(g^*+\ell)},$$

where g^* is the dual coxeter number of $\mathfrak{g}$ and ρ denotes the Weyl vector. If $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ is semisimple, we define the conformal anomaly and trace anomaly by taking sum of the conformal anomaly for each simple component.

Definition 3.1. *Let $\phi=(\phi_1,\phi_2):\mathfrak{s}=\mathfrak{g}_1\oplus\mathfrak{g}_2\rightarrow\mathfrak{g}$ be an embedding of Lie algebras with Dynkin-multi index $k=(k_1,k_2)$. We define ϕ to be a conformal embedding $\mathfrak{s}$ in $\mathfrak{g}$ at level ℓ if $c(\mathfrak{g}_1,k_1\ell)+c(\mathfrak{g}_2,k_2\ell)=c(\mathfrak{g},\ell)$.*

It is shown in [13] that the above equality only holds if $\ell=1$. Many familiar and important embeddings are conformal. For a complete list of conformal embedding we refer the reader to [2]. Next we list two important properties which makes conformal embeddings special.

(1) We recall (see [12]) that, since $\mathfrak{s}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ is semisimple, $\phi:\mathfrak{s}\rightarrow\mathfrak{g}$ is a conformal subalgebra is equivalent to the statement that any irreducible $\widehat{\mathfrak{g}}$-module $\mathcal{H}_{\Lambda}(\mathfrak{g})$ of level 1 decomposes into a finite sum of irreducible $\widehat{\mathfrak{s}}$-modules of level $\ell=(\ell_1,\ell_2)$.
(2) If $\phi:\mathfrak{s}\rightarrow\mathfrak{g}$ is a conformal embedding, then the action of the Virasoro operators are same i.e. for any n the following equality holds

$$L_n^{\mathfrak{s}}=L_n^{\mathfrak{g}}\in\operatorname{End}(\mathcal{H}_{\Lambda}(\mathfrak{g})),$$

where $L_n^{\mathfrak{s}}$ and $L_n^{\mathfrak{g}}$ are n-th Virasoro operators of $\mathfrak{s}$ and $\mathfrak{g}$ acting at level ℓ and 1 respectively. We refer the reader to [12] for more details.

3.2. **General context of Rank-level duality.** Consider a level 1 integrable highest weight $\widehat{\mathfrak{g}}$-module $\mathcal{H}_{\Lambda}(\mathfrak{g})$ and restrict it to $\widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2$. The module $\mathcal{H}_{\Lambda}(\mathfrak{g})$ decomposes into irreducible integrable $\widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2$ modules of level $\ell = (\ell_1, \ell_2)$ as follows:

$$\bigoplus_{(\lambda,\mu)\in B(\Lambda)} m^{\Lambda}_{\lambda,\mu}\mathcal{H}_{\lambda}(\mathfrak{g}_1) \otimes \mathcal{H}_{\mu}(\mathfrak{g}_2) \simeq \mathcal{H}_{\Lambda}(\mathfrak{g}),$$

where ℓ is the Dynkin multi index of ϕ and $m^{\Lambda}_{\lambda,\mu}$ is the multiplicity of the component $\mathcal{H}_{\lambda}(\mathfrak{g}_1) \otimes \mathcal{H}_{\mu}(\mathfrak{g}_2)$. Since the embedding is conformal, we know that both $|B(\Lambda)|$ and $m^{\Lambda}_{\lambda,\mu}$ are finite.

We consider only those conformal embeddings such that for every $\Lambda \in P_1(\mathfrak{g})$ and $(\lambda, \mu) \in B(\Lambda)$, the multiplicity $m^{\Lambda}_{\lambda,\mu} = 1$. Let $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ be a n-tuple of level 1 dominant weights of $\mathfrak{g}$. We consider $\mathcal{H}_{\vec{\Lambda}}(\mathfrak{g})$ and restrict it to $\widehat{\mathfrak{g}}_1 \oplus \widehat{\mathfrak{g}}_2$. Choose $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ such that $(\lambda_i, \mu_i) \in B(\Lambda_i)$ for all $1 \leq i \leq n$. We get a map

$$\bigotimes_{i=1}^{n}(\mathcal{H}_{\lambda_i}(\mathfrak{g}_1) \otimes \mathcal{H}_{\mu_i}(\mathfrak{g}_2)) \to \bigotimes_{i=1}^{n}\mathcal{H}_{\Lambda_i}(\mathfrak{g}).$$

Let $\mathfrak{X}$ denote the data associated to a curve C of genus g with n-distinct point $\vec{p} = (P_1, \ldots, P_n)$ with chosen coordinates $\xi_1, \ldots, \xi_n$. Taking coinvariants with respect to $\mathfrak{g}(\mathfrak{X})$ we get the following map:

$$\alpha : \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, \ell_1) \otimes \mathcal{V}_{\vec{\mu}}(\mathfrak{X}, \mathfrak{g}_2, \ell_2) \to \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{g}, 1),$$

If $\dim_{\mathbb{C}}(\mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{g}, 1)) = 1$, we get a map well defined up to constants

$$\alpha^{\vee} : \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, \ell_1) \to \mathcal{V}^{\dagger}_{\vec{\mu}}(\mathfrak{X}, \mathfrak{g}_2, \ell_2).$$

This map is known as the rank-level duality map.

Definition 3.2. *Let $\vec{\lambda} \in P^n_{\ell_1}(\mathfrak{g}_1)$ and $\vec{\mu} \in P^n_{\ell_2}(\mathfrak{g}_2)$. The pair $(\vec{\lambda}, \vec{\mu})$ is called admissible if one can define a rank-level duality map between the corresponding conformal blocks.*

Let $\mathcal{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_n; \xi_1, \ldots, \xi_n)$ be a family of nodal curves on a base $\mathcal{B}$ with sections s_i and local coordinates ξ_i. The rank-level duality map α can be easily extended to a map of sheaves

$$\alpha(\mathcal{F}) : \mathcal{V}_{\vec{\lambda}}(\mathcal{F}, \mathfrak{g}_1, \ell_1) \otimes \mathcal{V}_{\vec{\mu}}(\mathcal{F}, \mathfrak{g}_2, \ell_2) \to \mathcal{V}_{\vec{\Lambda}}(\mathcal{F}, \mathfrak{g}, 1),$$

3.3. **Properties of Rank-level duality.** In this section we recall some interesting properties of the rank-level duality maps. The following Proposition tell us about the behavior of the rank-level duality map in a smooth family of curves. For a proof see [6].

Proposition 3.3. *Let $\mathcal{F} = (\pi : \mathcal{C} \to \mathcal{B}; s_1, \ldots, s_n; \xi_1, \ldots, \xi_n)$ be a family of smooth projective curves on a base $\mathcal{B}$ with sections s_i and local coordinates ξ_i. Then the rank-level duality map α is projectively flat with respect to the $KZ/Hitchin/WZW$ connection.*

The rank-level duality map commutes with the propagation of vacua. The following has a direct proof.

Proposition 3.4. *Let Q be a point on the curve C distinct from $\vec{p} = (P_1, \ldots, P_n)$ and $\mathfrak{X}'$ be the data associated to the n-pointed curve. Consider $\vec{\lambda'} = (\lambda_1, \ldots, \lambda_n, 0)$ and $\vec{\mu'} = (\mu_1, \ldots, \mu_n, 0)$. The rank level duality map $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, \ell_1) \to \mathcal{V}^{\dagger}_{\vec{\mu}}(\mathfrak{X}, \mathfrak{g}_2, \ell_2)$ is an isomorphism if and only if the rank level duality map $\mathcal{V}_{\vec{\lambda'}}(\mathfrak{X}', \mathfrak{g}_1, \ell_1) \to \mathcal{V}^{\dagger}_{\vec{\mu'}}(\mathfrak{X}', \mathfrak{g}_2, \ell_2)$ is an isomorphism.*

3.3.1. *Diagram automorphisms and rank-level duality.* Let G be the complex simply connected group with Lie algebra $\mathfrak{g}$. The center $Z(G)$ acts as diagram automorphism of the affine Lie algebra $\widehat{\mathfrak{g}}$. The action of the center preserves the Cartan subalgebra of $\widehat{\mathfrak{g}}$ and hence it acts on $P_\ell(\mathfrak{g})$. Consider $\Gamma(G) = \{(\sigma_1, \ldots, \sigma_n) \in Z(G)^n | \prod_{i=1}^n \sigma_i = \mathrm{id}\}$ and for $\vec{\sigma} \in \Gamma(G)$ we denote by $\vec{\sigma}\vec{\lambda} = (\sigma_1\lambda_1, \ldots, \sigma_n\lambda_n)$. The following is one of the main results in [11].

Proposition 3.5. *Let $\mathfrak{X}$ be the data associated to n-distinct points with chosen coordinates on $\mathbb{P}^1$. Then there is an isomorphism*

$$\Theta : \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell) \to \mathcal{V}_{\vec{\sigma}\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}, \ell).$$

More over the isomorphism is flat with respect to the KZ/Hitchin connection.

The map Θ also have the following functorial property under embedding of Lie algebras. Let G_1, G_2 and G be simply connected Lie groups with simple Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}$ respectively. Consider a map $\phi : G_1 \times G_2 \to G$ and let $d\phi : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}$ be the map of Lie algebras. The proof of the following can be found in [15].

Proposition 3.6. *Let $\vec{\Sigma} \in \Gamma(G)$, $\vec{\sigma} \in \Gamma(G_1)$ be such that $\phi(\vec{\sigma}) = \vec{\Sigma}$, then the pairing*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, \ell_1) \otimes \mathcal{V}^{\dagger}_{\vec{\mu}}(\mathfrak{X}, \mathfrak{g}_2, \ell_2) \to \mathcal{V}^{\dagger}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{g}, 1)$$

is non degenerate if and only if the following pairing is non degenerate

$$\mathcal{V}_{\vec{\sigma}\vec{\lambda}}(\mathfrak{X}, \mathfrak{g}_1, \ell_1) \otimes \mathcal{V}^{\dagger}_{\vec{\mu}}(\mathfrak{X}, \mathfrak{g}_2, \ell_2) \to \mathcal{V}^{\dagger}_{\vec{\Sigma}\vec{\Lambda}}(\mathfrak{X}, \mathfrak{g}, 1).$$

4. Sewing and compatibility under factorization

First we recall the following Lemma from [21]. We refer the reader to [21] for the definition of the sheaf of conformal blocks. In this section we briefly recall a key compatibility theorem from [3]. Our notation is similar to [3].

Lemma 4.1. *There exists a bilinear pairing*

$$(\,,\,)_\lambda : \mathcal{H}_\lambda \times \mathcal{H}_{\lambda^\dagger} \to \mathbb{C}$$

unique up to a multiplicative constant such that

$$(X(n)u, v)_\lambda + (u, X(-n)v)_\lambda = 0.$$

for any $X \in \mathfrak{g}$, $n \in \mathbb{Z}$, $u \in \mathcal{H}_\lambda$ and $v \in \mathcal{H}_{\lambda^\dagger}$. Moreover the restriction of the form $(\,,\,)_\lambda$ to $\mathcal{H}_\lambda(m) \times \mathcal{H}_{\lambda^\dagger}(m')$ is zero if $m \neq m'$ and is non degenerate if $m = m'$.

Since the restriction of the bilinear form $(\,,\,)_\lambda$ to $\mathcal{H}_\lambda(m) \times \mathcal{H}_{\lambda^\dagger}(m)$ is non degenerate, this gives an isomorphism of $\mathcal{H}_{\lambda^\dagger}(m)$ with $\mathcal{H}_\lambda(m)^*$. Let $\gamma_\lambda(m)$ be the distinguished element of $\mathcal{H}_\lambda(m) \otimes \mathcal{H}_{\lambda^\dagger}(m)$ given by $(\,,\,)_\lambda$. Let t be a formal variable. We given $\lambda \in P_\ell(\mathfrak{g})$, we construct an element $\widetilde{\gamma}_\lambda = \sum_{m=0}^{\infty} \gamma_\lambda(m) t^m$ of $\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^\dagger}[[t]]$.

4.1. **Sewing.** We are now ready to describe the sewing procedure in [21]. Let $\mathcal{B} = \operatorname{Spec}\mathbb{C}[[t]]$. We consider a family of curves $\mathcal{X}$ over $\mathcal{B}$ such that its special fiber $\mathcal{X}_0$ is a curve X_0 over $\mathbb{C}$ with exactly one node and its generic fiber $\mathcal{X}_t$ is a smooth curve. Consider the sheaf of conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{X},\mathfrak{g})$ for the family of curves $\mathcal{X}$. The sheaf of conformal blocks commutes with base change and the fiber over any point $t \in \mathcal{B}$ coincides with $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}_t,\mathfrak{g})$, where $\mathfrak{X}_t$ is the data associated to the curve X_t over the point $t \in \mathcal{B}$.

Let $\widetilde{X}_0$ be the normalization of X_0. For $\lambda \in P_{\ell}(\mathfrak{g})$, the following isomorphism is constructed in [21]

$$\oplus\iota_{\lambda} : \bigoplus_{\lambda \in P_{\ell}(\mathfrak{g})} \mathcal{V}^{\dagger}_{\lambda,\lambda^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{X}},\mathfrak{g}) \to \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X},\mathfrak{g}),$$

where $\widetilde{\mathfrak{X}}$ is the data associated to the $(N+2)$-points of the smooth pointed curve $\widetilde{X}_0$.

In [21], a sheaf version of the above isomorphism is also proved. We briefly recall the construction. For every $\lambda \in P_{\ell}(\mathfrak{g})$ there exists a map

$$s_{\lambda} : \mathcal{V}^{\dagger}_{\lambda,\lambda^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{X}},\mathfrak{g}) \to \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{X},\mathfrak{g}),$$

where $s_{\lambda}(\psi) = \widetilde{\psi}$ and $\widetilde{\psi}(\widetilde{u}) := \psi(\widetilde{u} \otimes \widetilde{\gamma}_{\lambda}) \in \mathbb{C}[[t]]$ for any $\widetilde{u} \in \mathcal{H}_{\vec{\lambda}}[[t]]$. This map extends to a map $s_{\lambda}(t)$ of coherent sheaves of $\mathbb{C}[[t]]$-modules

$$s_{\lambda}(t) : \mathcal{V}^{\dagger}_{\lambda,\lambda^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{X}},\mathfrak{g}) \otimes \mathbb{C}[[t]] \to \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{X},\mathfrak{g}).$$

With the above notation, the following is proved in [21]

Proposition 4.2. *The map*

$$\oplus s_{\lambda}(t) : \bigoplus_{\lambda \in P_{\ell}(\mathfrak{g})} \mathcal{V}^{\dagger}_{\lambda,\lambda^{\dagger},\vec{\lambda}}(\widetilde{\mathfrak{X}},\mathfrak{g}) \otimes \mathbb{C}[[t]] \to \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{X},\mathfrak{g}).$$

is an isomorphism in a neighborhood of $t = 0$.

4.2. **Factorization and compatibility of rank-level duality.** Consider a conformal embedding of $\mathfrak{s} \to \mathfrak{g}$. Assume that all level one highest weight integrable modules of $\widetilde{\mathfrak{g}}$ decomposes with multiplicity one. As before let $\mathcal{X}$ be a family of curves of genus 0 over $\mathbb{C}[[t]]$ such that the generic fiber is a smooth curve and the special fiber is a nodal curve.

Let $\vec{\lambda} = (\lambda_1,\ldots,\lambda_n)$ be a n-tuple of level 1 weights of $\mathfrak{g}$ and $\vec{\mu} \in B(\vec{\lambda})$. We get a map $\mathcal{H}_{\vec{\mu}}(\mathfrak{s}) \to \mathcal{H}_{\vec{\lambda}}(\mathfrak{g})$. As discussed in Section 3, we get a $\mathbb{C}[[t]]$ linear map

$$\alpha(t) : \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{X},\mathfrak{g}) \to \mathcal{V}^{\dagger}_{\vec{\mu}}(\mathcal{X},\mathfrak{s}).$$

For $\mu \in B(\lambda)$, we denote by $\alpha_{\lambda,\mu}$ the rank level duality map

$$\mathcal{V}^{\dagger}_{\lambda,\lambda^{\dagger},\vec{\lambda}}(\widetilde{X}_0,\mathfrak{g}) \to \mathcal{V}^{\dagger}_{\mu,\mu^{\dagger},\vec{\mu}}(\widetilde{X}_0,\mathfrak{s})$$

and the extension of $\alpha_{\lambda,\mu}$ to a $\mathbb{C}[[t]]$ linear map

$$\alpha_{\lambda,\mu}(t) : \mathcal{V}^{\dagger}_{\lambda,\lambda^{\dagger},\vec{\lambda}}(\widetilde{X}_0,\mathfrak{g}) \otimes \mathbb{C}[[t]] \to \mathcal{V}^{\dagger}_{\mu,\mu^{\dagger},\vec{\mu}}(\widetilde{X}_0,\mathfrak{s}) \otimes \mathbb{C}[[t]]$$

The next Proposition from [3] describes how $\alpha(t)$ decomposes under factorization.

Proposition 4.3. *In a neighborhood of $t=0$ we have*

$$\alpha(t)\circ s_{\lambda}(t)=\sum_{\mu\in B(\lambda)} t^{n_{\mu}} s_{\mu}(t)\circ \alpha_{\lambda,\mu}(t),$$

where n_{μ} are integers depending on μ.

Remark 4.4. *The integers n_{μ} are non zero if the finite dimensional $\mathfrak{s}$ module V_{μ} does not appear in the decomposition of the finite dimensional $\mathfrak{g}$ module V_{λ}. This numbers n_{μ} controls the nature of the rank-level duality map on a nodal curve.*

Suppose V and W are vector bundles of same rank and L be a line bundle on $\operatorname{Spec}\mathbb{C}[[t]]$. Let f be a bilinear map from $V\otimes W\to L$. Assume that in a neighborhood of $t=0$, there are isomorphisms

$$\oplus s_i:\quad V\to \bigoplus_{i\in I} V_i$$

$$\oplus t_j:\quad W\to \bigoplus_{j\in I} W_j$$

Further assume that $s: L\to \widetilde{L}$ is an isomorphism. Let $f_{i,j}$ be maps from $V_i\otimes W_j\to\widetilde{L}$ such that $f_{i,j}=0$ for $i\neq j$ and $s\circ f=\sum_{i\in I} t^{m_i}(f_{i,i}\circ(s_i\otimes t_i))$. The following Lemma is easy to prove.

Corollary 4.5. *The map f is also non degenerate on $U^*=U\setminus\{t=0\}$ if and only if for all $i\in I$ the maps $f_{i,i}$'s are non degenerate, where U is a neighborhood of t.*

5. Branching Rules for conformal embedding of orthogonal Lie algebras

In this section, we discuss the branching rule for the conformal embedding $\mathfrak{so}(2r+1)\oplus\mathfrak{so}(2s+1)\to\mathfrak{so}((2r+1)(2s+1))$.

5.1. **Representation of $\mathfrak{so}(2r+1)$.** Let $E_{i,j}$ be a matrix whose i,j-th entry is 1 and all other entries are zero. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}(2r+1)$ is generated by diagonal matrices of the form $H_i=E_{i,i}-E_{r+i,r+i}$ for $1\leq i\leq r$. Let $L_i\in\mathfrak{h}^*$ be defined by $L_i(H_j)=\delta_{i,j}$. The normalized Cartan killing form on $\mathfrak{h}$ is given by $(H_i,H_j)=\delta_{ij}$. Under the identification of $\mathfrak{h}^*$ with $\mathfrak{h}$ using the Cartan Killing form the image of L_i is H_i for all $1\leq i\leq r$.

We can choose the simple positive roots of $\mathfrak{so}(2r+1)$ to be $\alpha_1=L_1-L_2,\alpha_2=L_2-L_3,\dots,\alpha_{r-1}=L_{r-1}-L_r,\alpha_r=L_r$. The highest root $\theta=L_1+L_2=\alpha_1+2\alpha_2+\cdots+2\alpha_r$. The fundamental weights of the B_n are $\omega_i=L_1+L_2+\cdots+L_i$ for $1\leq i<r$ and $\omega_r=\frac{1}{2}(L_1+L_2+\cdots+L_r)$.

The dominant integral weights P_+ of $\mathfrak{so}(2r+1)$ can be written as

$$P_+=P_+^0\sqcup P_+^1,$$

where P_+^0 is the set of dominant weight $\lambda=\sum_{i=1}^r a_i\omega_i$ such that a_r is even and $P_+^1:=P_+^0+\omega_r$. Let $\mathcal{Y}_r$ be the set of Young diagrams with at most r rows and $\mathcal{Y}_{r,s}$ denote the set of Young diagrams with at most r rows and s columns. Then P_+^0 is in bijection with $\mathcal{Y}_r$.

Combinatorially any dominant weight λ of P_+ can be written as $Y+t\omega_r$ where $t=\{0,1\}$ and $Y\in\mathcal{Y}$. If $t=0$, then $\lambda\in P_+^0$ and if $t=1$, then $\lambda\in P_+^1$.

Let $\lambda=\sum_{i=1}^{r}a_i\omega_i$ be a dominant integral weight. Then $(\theta,\lambda)=a_1+2(a_2+\cdots+a_{r-1})+a_r$. The set of level $2s+1$ dominant weights are described below:

$$P_{2s+1}(\mathfrak{so}(2r+1))=\{\lambda\in P_+|a_1+2(a_2+\cdots+a_{r-1})+a_r\leq 2s+1\}.$$

5.2. **Action of center on weights.** An element σ of the center of the group $\mathrm{Spin}(2r+1)$ acts as outer automorphisms on the affine Lie algebra $\widehat{\mathfrak{so}}(2r+1)$. For details we refer the reader to [13]. The action of σ on the $P_{2s+1}(\mathfrak{so}(2r+1))$ is given by $\sigma(\lambda)=(2s+1-(a_1+2(a_2+\cdots+a_{r-1})+a_r))\omega_1+a_2\omega_2+\cdots+a_r\omega_r$. We denote by $P^0_{2s+1}(\mathfrak{so}(2r+1))$ the intersection $P^0_+\cap P_{2s+1}(\mathfrak{so}(2r+1))$. The following Lemma can be proved by direct calculation.

Lemma 5.1. *The action of σ preserves the set $P^0_{2s+1}(\mathfrak{so}(2r+1))$ and $P^0_{2s+1}(\mathfrak{so}(2r+1))=\mathcal{Y}_{r,s}\sqcup\sigma(\mathcal{Y}_{r,s})$.*

We now describe the orbits of $P_{2s+1}(\mathfrak{so}(2r+1))$ under the action of the center following [17]. Let $\rho=\sum_{i=1}^{r}\omega_i$ be the Weyl vector. For $\lambda=\sum_{i=1}^{r}a_i\omega_i$, the weight $\lambda+\rho=\sum_{i=1}^{r}t_i\omega_i$, where $t_i=a_i+1$. Put $u_i=\sum_{j=i}^{r-1}t_j+\frac{t_r}{2}$ for $1\leq i\leq r$, $u_r=\frac{t_r}{2}$ and $u_{r+1}=0$.

The set $P_{2s+1}(\mathfrak{so}(2r+1))$ gets identifies with the collection of sets $U=(u_1>u_2>\cdots>u_r>0)$ such that

- $u_i\in\frac{1}{2}\mathbb{Z}$.
- $u_i-u_{i+1}\in\mathbb{Z}$.
- $u_1+u_2\leq 2(r+s)$.

Let $P^+_{2s+1}(\mathfrak{so}(2r+1))$ denote the set of weights in $P_{2s+1}(\mathfrak{so}(2r+1))$ such that $u_i\in\mathbb{Z}$.

Let us set $k=2(r+s)$. We now describe the action of the center Γ on $P_{2s+1}(\mathfrak{so}(2r+1))$ in as exchanging t_1 with $t_0=k-t_1-2t_2-\cdots-2t_{r-1}-t_r$ or in other words changing u_1 with $k-u_1$. We observe that the action of Γ preserves $P^+_{2s+1}(\mathfrak{so}(2r+1))$ and $P^0_{2s+1}(\mathfrak{so}(2r+1))$. Then we can identify the orbits of the action of Γ as follows :

$$P_{2s+1}(\mathfrak{so}(2r+1))/\Gamma=\{U=(u_1,\dots,u_r)|\frac{k}{2}\geq u_1>u_2>\cdots>u_r>0,u_i\in\frac{1}{2}\mathbb{Z},u_i-u_{i+1}\in\mathbb{Z}\}.$$

and the length of the orbits is given as follows:

- $|\Gamma(U)|=2$ if $u_1<\frac{k}{2}$.
- $|\Gamma(U)|=1$ if $u_1=\frac{k}{2}$.

For any number a and a set $U=(u_1>u_2>\cdots>u_r)$ denote by $U-a$ and $a-U$, the set $\{u_1-a>u_2-a>\cdots>u_r-a\}$ and $\{a-u_r>a-u_{r-1}>\cdots>a-u_1\}$ respectively. The following two Lemmas from [17] gives a bijection of orbits.

Lemma 5.2. *Let $P_{2r+1}(\mathfrak{so}(2s+1))$ denote the weights of $\mathfrak{so}(2s+1)$ of level $2r+1$. Then there is a bijection between the orbits of $P^+_{2s+1}(\mathfrak{so}(2s+1))$ and the orbit of $P^+_{2r+1}(\mathfrak{so}(2s+1))$ given by*

$$U=(u_1>u_2>\cdots>u_r)\to U^c=(u_1^c>\cdots>u_s^c)$$

where $U\subset[r+s]$ of cardinality r and U^c is the complement of U in $[r+s]$.

For $\lambda\in P^0_{2s+1}(\mathfrak{so}(2r+1))$. We write $\lambda+\rho=\sum_{i=1}^{r}(u'_i-\frac{1}{2})L_i$, where u'_i are all integers. We identify the identify the orbits of $P^0_{2s+1}(\mathfrak{so}(2\mathfrak{r}+1))$ under Γ as subsets $U'=(u'_1>u'_2>\cdots>u'_r)$ of $[r+s]$.

Lemma 5.3. *There is a bijection between the orbits of $P^0_{2s+1}(\mathfrak{so}(2s+1))$ and the orbits of $P^0_{2r+1}(\mathfrak{so}(2s+1))$ given by*

$$U' = (u'_1 > u'_2 > \cdots > u'_r) \to ((r+s)+1-U'^c) = (u''_1 > \cdots > u''_s)$$

where $U' \subset [r+s]$ of cardinality r and U'^c is the complement of U' in $[r+s]$.

5.3. **Branching rules.** We now describe the branching rules for the conformal embedding $\mathfrak{so}(2r+1) \oplus \mathfrak{so}(2s+1) \subset \mathfrak{so}((2r+1)(2s+1))$. Let $N = (2r+1)(2s+1) = 2d+1$. The level one highest weights of $\widehat{\mathfrak{so}}(N)$ are 0, ω_1 and ω_d. The decompositions of the level one integrable highest weight modules of weight 0 and ω_1 are given in the following Proposition. For a proof we refer the reader to [13].

Proposition 5.4. *Let $\mathcal{H}_0(\mathfrak{so}(N))$ and $\mathcal{H}_1(\mathfrak{so}(N))$ denote the highest weight integrable modules of the affine Lie algebra with highest weight 0 and ω_1 respectively. Then the module $\mathcal{H} := \mathcal{H}_0(\mathfrak{so}(N)) \oplus \mathcal{H}_1(\mathfrak{so}(N))$ breaks up as a direct sum of highest weight integrable module of $\widehat{\mathfrak{so}}(2r+1) \oplus \widehat{\mathfrak{so}}(2s+1)$ of the form*

- $\mathcal{H}_\lambda(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$.
- $\mathcal{H}_\lambda(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\sigma\lambda^T}(\mathfrak{so}(2s+1))$.
- $\mathcal{H}_{\sigma\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$.
- $\mathcal{H}_{\sigma\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\sigma\lambda^T}(\mathfrak{so}(2s+1))$.

where $\lambda \in \mathcal{Y}_{r,s}$ and σ is an automorphism associated to the center of the simply connected group. More over all of the above factors appear with multiplicity 1.

We need to determine which factor in the above decomposition rules comes from $\mathcal{H}_0(\mathfrak{so}(N)$ and which factor comes from $\mathcal{H}_1(\mathfrak{so}(N))$.

Lemma 5.5.

$$\Delta_0(\mathfrak{so}(N)) = 0 \quad \Delta_{\omega_1}(\mathfrak{so}(N)) = \frac{1}{2}$$

Lemma 5.6. *For $\lambda \in \mathcal{Y}_{r,s}$ we have the following equality*

$$\Delta_\lambda(\mathfrak{so}(2r+1)) + \Delta_{\lambda^T}(\mathfrak{so}(2s+1)) = \frac{1}{2}|\lambda|$$

Corollary 5.7. *Let $\mathcal{H}_0(\mathfrak{so}(N))$ denote the level 1 highest weight integrable $\widehat{\mathfrak{so}}(N)$-module of weight 0. Then the following factors appears as the decomposition of $\mathcal{H}_0(\mathfrak{so}(N))$ as $\widehat{\mathfrak{so}}(2r+1) \oplus \widehat{\mathfrak{so}}(2s+1)$ modules.*

- $\mathcal{H}_\lambda(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is even.*
- $\mathcal{H}_{\sigma\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\sigma\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is even.*
- $\mathcal{H}_{\sigma\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is odd.*
- $\mathcal{H}_\lambda(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\sigma\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is odd.*

Corollary 5.8. *Let $\mathcal{H}_1(\mathfrak{so}(N))$ denote the level 1 highest weight integrable $\widehat{\mathfrak{so}}(N)$-module of weight ω_1. Then the following factors appears as the decomposition of $\mathcal{H}_0(\mathfrak{so}(N))$ as $\widehat{\mathfrak{so}}(2r+1) \oplus \widehat{\mathfrak{so}}(2s+1)$ modules.*

- $\mathcal{H}_\lambda(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is odd.*
- $\mathcal{H}_{\sigma\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\sigma\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is odd.*
- $\mathcal{H}_{\sigma\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$, *when $|\lambda|$ is even.*

- $\mathcal{H}_{\lambda}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\sigma\lambda^T}(\mathfrak{so}(2s+1))$, *when* $|\lambda|$ *is even.*

Remark 5.9. *Except* $\lambda = 0$ *and* $\lambda = \omega_1$, *all the components that appear in the above decomposition does not appear in the decomposition of standard and trivial representations of the finite dimensional Lie algebra* $\mathfrak{so}(2d+1)$ *into* $\mathfrak{so}(2r+1) \oplus \mathfrak{so}(2s+1)$*-modules. This is the main obstruction to construct the rank-level duality map in Theorem 1.1 geometrically. It is important to study this map geometrically to understand rank-level duality on curves of higher genus. This will be considered in a subsequent paper.*

6. Dimensions of some conformal blocks

In this section, we calculate the dimension of some conformal blocks. Let $\mathfrak{g}$ be any simple Lie algebra and $\mathfrak{s}_\theta$ denote the Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$ generated by H_θ, $\mathfrak{g}_\theta$ and $\mathfrak{g}_{-\theta}$. A $\mathfrak{g}$-module V of level ℓ decomposes as a direct sum of $\mathfrak{s}_\theta$ modules as follows:

$$V \simeq \oplus_{i=1}^{\ell} V^i$$

where V^i is a direct sum of $\mathfrak{sl}_2$ modules isomorphic to $\mathrm{Sym}^i\,\mathbb{C}^2$. We recall the following description of conformal blocks on 3-pointed $\mathbb{P}^1$ from [4].

Proposition 6.1. *Let* $\mathfrak{X}$ *be the data associated to the* 3*-pointed* $\mathbb{P}^1$ *and* $\lambda, \mu, \nu \in P_\ell(\mathfrak{g})$ *with chosen coordinates. Then the conformal block* $\mathcal{V}^\dagger_{\lambda,\mu,\nu}(\mathfrak{X}, \mathfrak{g})$ *is canonically isomorphic to the space of* $\mathfrak{g}$ *invariant forms* ϕ *on* $V_\lambda \otimes V_\mu \otimes V_\nu$ *such that the* ϕ *restricted to* $V^p_\lambda \otimes V^q_\mu \otimes V^r_\nu$ *is zero when ever* $p+q+r > 2\ell$.

6.1. **The case $\mathfrak{g} = \mathfrak{so}(2r+1)$ with level 1.** Let $\vec{p} = (P_1, P_2, P_3)$ be three distinct points on $\mathbb{P}^1$ with chosen coordinates and $\mathfrak{X}$ be the associated data. The level one dominant integral weights of $\mathfrak{so}(2r+1)$ are 0, ω_1 and ω_r. Let $\mathcal{V}^\dagger_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{X}, \mathfrak{so}(2r+1))$ denote the conformal blocks on $\mathbb{P}^1$ with three marked points and weights $\lambda_1, \lambda_2, \lambda_3$ at level 1. The following are proved in [7]:

- $\dim_{\mathbb{C}} \mathcal{V}^\dagger_{\omega_1,\omega_1,0}(\mathfrak{X}, \mathfrak{so}(2r+1)) = 1$
- $\dim_{\mathbb{C}} \mathcal{V}^\dagger_{\omega_1,\omega_1,\omega_1}(\mathfrak{X}, \mathfrak{so}(2r+1)) = 0$
- $\dim_{\mathbb{C}} \mathcal{V}^\dagger_{\omega_1,\omega_1,\omega_r}(\mathfrak{X}, \mathfrak{so}(2r+1)) = 0$
- $\dim_{\mathbb{C}} \mathcal{V}^\dagger_{\omega_1,\omega_r,\omega_r}(\mathfrak{X}, \mathfrak{so}(2r+1)) = 1$

Lemma 6.2. *Let* $P_1, \dots P_n$ *be* n *distinct points on* $\mathbb{P}^1$ *with chosen coordinates and* $\mathfrak{X}$ *be the associated data. Assume that* $\vec{\lambda} = (\omega_1, \dots, \omega_1)$. *Then* $\dim_{\mathbb{C}} \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 1) = 1$ *if* n *is even and zero if* n *is odd.*

Proof. The proof follows from above and factorization of conformal blocks. □

6.2. **The case $\mathfrak{g} = \mathfrak{so}(2r+1)$ at level ℓ.** We calculate the dimension of some special conformal blocks on 3 pointed $\mathbb{P}^1$ at any level ℓ. We first recall the following tensor product decomposition from [14]:

Proposition 6.3. *Let* $\lambda = \sum_{i=1}^{r} a_i\omega_i \in P^0_+$. *The Littlewood-Richardson rule for the tensor product decomposition of* $V_\lambda \otimes V_{\omega_1}$ *is given as follows:*

$$V_\lambda \otimes V_{\omega_1} \simeq \oplus_\gamma V_\gamma,$$

where γ is either λ if $a_r \neq 0$ or is obtained from λ by adding or deleting a box of the Young diagram of λ.

We use the above Proposition to calculate the dimension of the following conformal blocks.

Proposition 6.4. *Let $\lambda = \sum_{i=1}^{r} a_i\omega_i \in P_+^0$. Assume that $\lambda \in P_\ell(\mathfrak{so}(2r+1))$. Then the dimension of the conformal block $\mathcal{V}^{\dagger}_{\lambda,\gamma,\omega_1}(\mathfrak{X}, \mathfrak{so}(2r+1))$ of level ℓ is 1 where γ is either λ if $a_r \neq 0$ or is obtained from λ by adding or deleting a box of the the Young diagram of λ and 0 otherwise.*

Proof. The otherwise part follows directly from Proposition 6.3. Assume that $a_r \neq 0$ and γ is either λ or obtained from λ by adding or deleting a box. For a $\mathfrak{so}(2r+1)$ equivariant form ϕ on $V_\lambda \otimes V_{\omega_1} \otimes V_\gamma$, it's restriction to $V^1_{\omega_1} \otimes V^\ell_\lambda \otimes V^\ell_\gamma$ is zero, since $\mathbb{C}^2 \otimes \operatorname{Sym}^\ell \mathbb{C}^2$ does not contain $\operatorname{Sym}^\ell \mathbb{C}^2$ as an $\mathfrak{sl}_2(\mathbb{C})$-submodule. Thus by Proposition 6.1, the dimension of $\mathcal{V}^{\dagger}_{\lambda,\gamma,\omega_1}(\mathfrak{X}, \mathfrak{so}(2r+1))$ is one. The case when $a_r = 0$ follows similarly. □

7. Rank level duality map.

In this section using the branching rule we describe the rank level duality map. We consider the following weights:

- $\vec{\lambda}_i = (\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{n_1}})$ and $\vec{\lambda}_i^T = (\lambda_{i_1}^T, \lambda_{i_2}^T, \ldots, \lambda_{i_{n_1}}^T)$ where $\lambda_{i_a} \in \mathcal{Y}_{r,s}$ such that $|\lambda_{i_a}|$ is odd for each $1 \leq a \leq n_1$.
- $\vec{\lambda}_j = (\sigma\lambda_{j_1}, \ldots, \sigma\lambda_{j_{n_2}})$ and $\vec{\lambda}_j^T = (\sigma(\lambda_{j_1}^T), \ldots, \sigma(\lambda_{j_{n_2}}^T))$, where $\lambda_{j_a} \in \mathcal{Y}_{r,s}$ such that $|\lambda_{j_a}|$ is odd for all $1 \leq a \leq n_2$
- $\vec{\lambda}_k = (\sigma\lambda_{k_1}, \ldots, \sigma\lambda_{k_{n_3}})$ and $\vec{\lambda}_k^T = (\lambda_{k_1}^T, \ldots, \lambda_{k_{n_3}}^T)$, where $\lambda_{j_a} \in \mathcal{Y}_{r,s}$ such that $|\lambda_{k_a}|$ is even for all $1 \leq a \leq n_3$.
- $\vec{\lambda}_l = (\lambda_{l_1}, \lambda_{l_2}, \ldots, \lambda_{l_{n_4}})$ and $\vec{\lambda}_l^T = (\sigma(\lambda_{l_1}^T), \ldots, \sigma(\lambda_{l_{n_4}}^T))$ where $\lambda_{l_a} \in \mathcal{Y}_{r,s}$ such that $|\lambda_{l_a}|$ is even for each $1 \leq a \leq n_4$.
- $\vec{\beta}_i = (\beta_{i_1}, \ldots, \beta_{i_{m_1}})$ and $\vec{\beta}_i^T = (\beta_{i_1}^T, \beta_{i_2}^T, \ldots, \beta_{i_{m_1}}^T)$ where $\lambda_{i_a} \in \mathcal{Y}_{r,s}$ such that $|\lambda_{i_a}|$ is even for each $1 \leq a \leq m_1$.
- $\vec{\beta}_j = (\sigma\beta_{j_1}, \ldots, \sigma\beta_{j_{m_2}})$ and $\vec{\beta}_j^T = (\sigma(\beta_{j_1}^T), \ldots, \sigma(\beta_{j_{m_2}}^T))$, where $\beta_{j_a} \in \mathcal{Y}_{r,s}$ such that $|\beta_{j_a}|$ is even for all $1 \leq a \leq m_2$
- $\vec{\beta}_k = (\sigma\beta_{k_1}, \ldots, \sigma\beta_{k_{m_3}})$ and $\vec{\beta}_k^T = (\beta_{k_1}^T, \ldots, \beta_{k_{m_3}}^T)$, where $\lambda_{j_a} \in \mathcal{Y}_{r,s}$ such that $|\beta_{k_a}|$ is odd for all $1 \leq a \leq m_3$.
- $\vec{\beta}_l = (\beta_{l_1}, \beta_{l_2}, \ldots, \beta_{l_{m_4}})$ and $\vec{\beta}_l^T = (\sigma(\beta_{l_1}^T), \ldots, \sigma(\beta_{l_{m_4}}^T))$ where $\beta_{l_a} \in \mathcal{Y}_{r,s}$ such that $|\beta_{l_a}|$ is odd for each $1 \leq a \leq m_4$.

Let $n = \sum_{i=1}^{4}(n_i + m_i)$ be a positive integer, $\vec{\lambda} = \vec{\lambda}_i \cup \vec{\lambda}_j \cup \vec{\lambda}_k \cup \vec{\lambda}_l$, $\vec{\lambda}^T = \vec{\lambda}_i^T \cup \vec{\lambda}_j^T \cup \vec{\lambda}_k^T \cup \vec{\lambda}_l^T$, $\vec{\beta} = \vec{\beta}_i \cup \vec{\beta}_j \cup \vec{\beta}_k \cup \vec{\beta}_l$, $\vec{\beta}^T = \vec{\beta}_i^T \cup \vec{\beta}_j \cup \vec{\beta}_k^T \cup \vec{\beta}_l^T$ and $\mathfrak{X}$ be the data associated to n distinct points on $\mathbb{P}^1$ with chosen coordinates. Then we have the following map between conformal blocks

$$\alpha : \mathcal{V}_{\vec{\lambda}\cup\vec{\beta}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \otimes \mathcal{V}_{\vec{\lambda}^T\cup\vec{\beta}^T}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1) \to \mathcal{V}_{\vec{\omega_1}\cup\vec{\omega}_0}(\mathfrak{X}, \mathfrak{so}(N), 1),$$

where $\vec{\omega}_1 = (\omega_1, \ldots, \omega_1)$ is $(n_1+n_2+n_3+n_4)$-tuple of ω_1's and $\vec{0} = (0, \ldots, 0)$ be a $(m_1 + m_2 + m_3 + m_4)$-tuple of 0's.

Assume that $(n_1+n_2+n_3+n_4)$ is even, then $\dim_{\mathbb{C}} \mathcal{V}_{\vec{\omega}_1\cup\vec{0}}(\mathfrak{X},\mathfrak{so}(N),1)=1$. Then we have the following map:

$$\alpha^{\vee}:\mathcal{V}_{\vec{\lambda}\cup\vec{\beta}}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1)\rightarrow \mathcal{V}^{\dagger}_{\vec{\lambda}^T\cup\vec{\beta}^T}(\mathfrak{X},\mathfrak{so}(2s+1),2r+1) \tag{7.1}$$

This map $\alpha^{\vee}$ is called the rank-level duality map. The main result of this paper is the following:

Theorem 7.1. *The rank level-duality map defined above is an isomorphism.*

The rest of the paper is devoted to the proof of Theorem 7.1. First we observe that by Proposition 3.6 and Proposition 3.4, we can reduce the statement of Theorem 7.1 into the following non-equivalent statements. We will use these to check that the source and the target of the rank-level duality maps are same.

(1) If $\sum_{i=1}^n|\lambda_i|$ is even, then

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1)\simeq\mathcal{V}^{\dagger}_{\vec{\lambda}^T}(\mathfrak{X},\mathfrak{so}(2s+1),2r+1)$$

(2) If $\sum_{i=1}^n|\lambda_i|$ is odd, then

$$\mathcal{V}_{\vec{\lambda},0}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1)\simeq\mathcal{V}^{\dagger}_{\vec{\lambda}^T,\sigma(0)}(\mathfrak{X},\mathfrak{so}(2s+1),2r+1)$$

(3) If $\sum_{i=1}^n|\lambda_i|$ is even, then

$$\mathcal{V}_{\vec{\lambda},\sigma(0)}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1,\vec{z})\simeq\mathcal{V}^{\dagger}_{\vec{\lambda}^T,\sigma(0)}(\mathfrak{X},\mathfrak{so}(2s+1),2r+1)$$

Remark 7.2. *The decomposition of the highest integrable module of $\widehat{\mathfrak{so}}(N)$ with highest weight ω_d is also given in* [13]*. Also the decomposition of level highest weight integrable for all orthogonal Lie algebras are given in* [13]*. A similar study of rank-level duality questions for the other cases will be done in a subsequent paper.*

8. Equality of dimensions

In this section we prove that the dimensions of source and the target of Theorem 7.1 are same.

8.1. **Weyl Character formula.** Here we first state a basic matrix identity which is an easy generalization of a Lemma A.42 from [10]. If $A=(a_{ij})$ is an $(r+s)\times(r+s)$ matrix and $U=(u_1,\dots,u_r)$ and $T=(t_1,\dots,t_r)$ be two sequence of r distinct integers from $\{1,2,\dots,(r+s)\}$. Let $A_{U,T}$ denote the $r\times r$ matrix whose i,j-th entry is a_{u_i,t_j} Similarly define the $s\times s$ matrix B_{U^c,T^c} where U^c and T^c are the complement of U and T.

Lemma 8.1. *Let A and B be two $(r+s)\times(r+s)$ matrices whose product is a diagonal matrix D whose (i,i)-th entry is a_i. Let $\pi=(U,U^c)$ and (T,T^c) be permutations of the sequence $(1,\dots,r+s)$ where $|U|=|T|=r$. Then*

$$\left(a_{\pi(r+1)}\dots a_{\pi(r+s)}\right)\det A_{U,T}=\operatorname{sgn}(U,U^c)\operatorname{sgn}(T,T^c)\det A\det B_{T^c,U^c}$$

Proof. Consider the permutation matrix P, Q^{-1} associated to the permutation (U,U^c) and (T,T^c) respectively. Then

$$PAQ=\begin{pmatrix}A_1 & A_2\\ A_3 & A_4\end{pmatrix}\text{ where } A_{U,T}=A_1$$

and similarly

$$Q^{-1}BP^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \text{ where } B_{T^c,U^c} = B_4$$

Now

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \times \begin{pmatrix} I_k & B_2 \\ 0 & B_4 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ A_3 & \Lambda \end{pmatrix}$$

where Λ is a diagonal matrix whose (i,i)-th entry is $a_{\pi(r+i)}$. Taking determinant of both sides of the above matrix equation we get the desired equality. $\square$

We are now ready to state the Weyl character formula for $\mathfrak{so}(2r+1)$ following [10]. Let $\mu \in P^0_{2s+1}(\mathfrak{so}(2r+1))$ and $\mu+\rho = \sum_{i=1}^r u_i L_i$ where u_i is as defined in a Section 5. Let $\lambda = \sum_{i=1}^r \lambda^i \mathrm{L}_i$ be any dominant weight of $\mathfrak{so}(2r+1)$ and V_λ be the irreducible highest weight module of $\mathfrak{so}(2r+1)$ with weight λ. Then by the Weyl character formula

$$\mathrm{Tr}_{V_\lambda}(\exp \pi\sqrt{-1}\frac{\mu+\rho}{(r+s)}) = \frac{\det\left(\zeta^{u_i(\lambda_j+r-j+\frac{1}{2})} - \zeta^{-u_i(\lambda_j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{u_i(r-j+\frac{1}{2})} - \zeta^{-u_i(r-j+\frac{1}{2})}\right)},$$

where $\mu+\rho$ is considered as element of $\mathfrak{h}$ under the identification of $\mathfrak{h}$ with $\mathfrak{h}^*$, exp is the exponential map from $\mathfrak{so}(2r+1)$ to SO_{2r+1}, $\zeta = \exp\left(\frac{\pi\sqrt{-1}}{r+s}\right)$.

8.2. **Verlinde Formula.** Let us first recall the Verlinde formula in full generality. Let C be a nodal curve of genus g and $P_1,\ldots,P_n$ be n distinct smooth points on C and $\mathfrak{X}$ be the associated data. We fix a Lie algebra $\mathfrak{g}$, and $\vec{\lambda} = (\lambda_1,\lambda_2,\ldots,\lambda_n)$ be an n-tuple of dominant weight of $\mathfrak{g}$ of level ℓ. We refer the reader to [4], [8], [21] for a proof.

Theorem 8.2. *The dimension of the conformal block $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X},\mathfrak{g})$ is given by the following formula*

$$\{(\ell+g^*)^{\mathrm{rank}\,\mathfrak{g}}|P/Q_{long}|\}^{g-1}\sum_{\mu\in P_\ell(\mathfrak{g})}\mathrm{Tr}_{V_{\vec{\lambda}}}(\exp 2\pi\sqrt{-1}\frac{\mu+\rho}{\ell+g^*})\prod_{\alpha>0}\left|2\sin\pi\frac{(\mu+\rho,\alpha)}{\ell+g^*}\right|^{2-2g},$$

where exp *is the exponential map form $\mathfrak{g}$ to the simple connected Lie group G, Q_{long} is the lattice of long roots and g^* is the dual coxeter number of $\mathfrak{g}$.*

Let us now specialize to the case $g=0$, $\mathfrak{g}=\mathfrak{so}(2r+1)$, $\ell = 2s+1$ and $\vec{\lambda} = (\lambda_1,\ldots,\lambda_n)$ be an n-tuple of weight in $P^0_{2s+1}(\mathfrak{so}(2r+1))$. The dual coxeter number $\mathfrak{so}(2r+1)$ is $2r-1$ and $\{(\ell+g^*)^{\mathrm{rank}\,\mathfrak{g}}|P/Q_{long}|\} = 4(k)^r$, where $k=2(r+s)$. Then the dimension of the conformal block is given by

$$\sum_{U\in P_{2s+1}(\mathfrak{so}(2r+1))}\prod_{q=1}^{n}\frac{\det\left(\zeta^{u_i(\lambda^j_q+r-j+\frac{1}{2})} - \zeta^{-u_i(\lambda^j_q+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{u_i(r-j+\frac{1}{2})} - \zeta^{-u_i(r-j+\frac{1}{2})}\right)}\left(\frac{\Phi_k(U)}{4k^r}\right), \tag{8.1}$$

where $\mu+\rho = \sum_{i=1}^r u_i L_i$, the set $U = (u_1 > u_2 > \cdots > u_r)$, $\lambda_q = (\lambda^1_q, \lambda^2_q,\ldots,\lambda^r_q)$ and $\Phi_k(U)$ as in Section 2 of [17].

Lemma 8.3. *Let σ be the nontrivial element of the center of* $\mathrm{Spin}(2r+1)$ *that acts by diagram automorphism on the level $2s+1$ weights of $\mathfrak{so}(2r+1)$. Then*

$$\mathrm{Tr}_{V_{\vec{\lambda}}}(\exp\pi\sqrt{-1}\frac{\sigma\mu+\rho}{r+s}) = \mathrm{Tr}_{V_{\vec{\lambda}}}(\exp\pi\sqrt{-1}\frac{\mu+\rho}{r+s})$$

Proof. Let $\mu=\sum_{i=1}^{r}a_i\omega_i\in P_{2s+1}(\mathfrak{so}(2r+1))$. Then the weight $\sigma(\mu)$ is given by the formula $(2s+1-2(a_1+\cdots+a_r)+a_1+a_r)\omega_1+\sum_{i=2}^{r}a_i\omega_i$. Then we have the following;

$$\begin{aligned}\sigma(\mu)+\rho &= (2s+2-2(a_1+\cdots+a_r)+a_1+a_r)\omega_1+\sum_{i=2}^{r}(a_i+1)\omega_i,\\ &= ((2s+1)-(a_1+\cdots+a_{r-1})-\frac{a_r}{2}+\frac{2r-1}{2})L_1+\\ &\quad ((a_2+a_3+\cdots+a_{r-1})+(r-2)+\frac{a_r+1}{2})L_2+\cdots+\frac{a_r+1}{2}L_r.\end{aligned}$$

Let w be a element of the Weyl group of $\mathfrak{so}(2r+1)$ which send $L_1\to -L_1$. Then

$$\begin{aligned}w.(\sigma\mu+\rho) &= (a_1+a_2+\cdots+\frac{a_r}{2}-(2s+1)-(2r-1)+r-\frac{1}{2})L_1+\\ &\quad ((a_2+a_3+\cdots+a_{r-1})+(r-2)+\frac{a_r+1}{2})L_2+\cdots+\frac{a_r+1}{2}L_r,\\ &= \mu+\rho-2(r+s)L_1.\end{aligned}$$

We taking exponential map to get the following identity.

$$\exp(2\pi\sqrt{-1}\frac{w.(\sigma\mu+\rho)}{2(r+s)}) \;=\; \exp(2\pi\sqrt{-1}\frac{\mu+\rho}{2(r+s)})$$

Hence the Lemma follows by the Weyl character formula. □

Let $\mu+\rho=\sum_{i=1}^{r}u_iL_i$ and $\mu'+\rho'=\sum_{i=1}^{s}(u'_i-\frac{1}{2})L_i$. Consider the sets $U=(u_1>u_2>\cdots>u_r)$, and $U'=(u'_1>\cdots>u'_r)$ and let $[U]$ and $[U']$ denote the class of μ, μ' in $P^{+}_{2s+1}(\mathfrak{so}(2r+1))/\Gamma$ and $P^{0}_{2s+1}(\mathfrak{so}(2r+1))/\Gamma$ respectively.

Without loss of generality we can assume that $V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n}$ has non-trivial invariants since this is a necessary condition for the conformal block to be non-zero. Since the function Φ_k is invariant under the action of center, by Lemma 8.3 we can rewrite the Verlinde formula in 8.1 in the following way:

$$(8.2)\quad \sum_{[U]\in P^{+}_{2s+1}(\mathfrak{so}(2r+1))/\Gamma}|\,\mathrm{Orb}_U\,|\prod_{q=1}^{n}\frac{\det\left(\zeta^{u_i(\lambda^j_q+r-j+\frac{1}{2})}-\zeta^{-u_i(\lambda^j_q+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{u_i(r-j+\frac{1}{2})}-\zeta^{-u_i(r-j+\frac{1}{2})}\right)}\left(\frac{\Phi_k(U)}{4k^r}\right)+$$

$$(8.3)\quad \sum_{[U']\in P^{0}_{2s+1}(\mathfrak{so}(2r+1))/\Gamma}|\,\mathrm{Orb}_{U'}\,|\prod_{q=1}^{n}\frac{\det\left(\zeta^{(u'_i-\frac{1}{2})(\lambda^j_q+r-j+\frac{1}{2})}-\zeta^{-(u'_j-\frac{1}{2})(\lambda^j_q+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{(u'_i-\frac{1}{2})(r-j+\frac{1}{2})}-\zeta^{-(u'_i-\frac{1}{2})(r-j+\frac{1}{2})}\right)}\left(\frac{\Phi_k(U'-\frac{1}{2})}{4k^r}\right),$$

where $|\,\mathrm{Orb}_U\,|$, $|\,\mathrm{Orb}_{U'}\,|$ denote the length of the orbits of μ and μ' under the action of Γ on $P^{+}_{2s+1}(\mathfrak{so}(2r+1))$ and $P^{0}_{2s+1}(\mathfrak{so}(2r+1))$. The sets $P^{+}_{2s+1}(\mathfrak{so}(2r+1))/\Gamma$ and $P^{0}_{2s+1}(\mathfrak{so}(2r+1))/\Gamma$ are the orbits of $P^{+}_{2s+1}(\mathfrak{so}(2r+1))$ and $P^{0}_{2s+1}(\mathfrak{so}(2r+1))$ under the action of Γ.

8.3. **Final Step of Dimension check.** The main result of this section is that using the Verlinde formula we prove that the target and the source of the rank level duality map in Theorem 7.1 have same dimension.

Let us recall the following two lemmas from [17]. We refer the reader to [17], pages 2694-2695 for a proof.

Lemma 8.4. *For a positive integer a, let V and V^c be complementary subsets of $\{1,\dots,a-1\}$. Then*

$$\frac{(2a)^{|V|}}{\Phi_{2a}(V)}=\frac{2(2a)^{|V^c\cup\{a\}|}}{\Phi_{2a}(V^c\cup\{a\})}$$

Lemma 8.5. *Let $V'\subset S=\{\frac{1}{2},\dots a-\frac{1}{2}\}$ and V'^c be the complement. Then we have:*

$$\frac{(2a)^{|V'|}}{\Phi_{2a}(V')}=\frac{(2a)^{V'^c}}{\Phi_{2a}(a-V'^c)}$$

Let $\lambda_i\in\mathcal{Y}_{r,s}$ such that $\sum_{i=1}^n|\lambda_i|$ is even and $\mathfrak{X}$ be the data associated to n-distinct points on $\mathbb{P}^1$. Denote by $\vec{\lambda}$ the n-tuple of weights $(\lambda_1,\dots,\lambda_n)$ and $\vec{\lambda}^T$ the n-tuple of weights $(\lambda_1^T,\dots,\lambda_n^T)$. Consider the conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X},\mathfrak{so}(2r+1))$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}^T}(\mathfrak{X},\mathfrak{so}(2s+1))$.

Proposition 8.6. *If $\sum_{i=1}^n|\lambda|$ is even, then the following equality of dimensions holds.*

$$\dim_{\mathbb{C}}\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1)=\dim_{\mathbb{C}}\mathcal{V}^{\dagger}_{\vec{\lambda}^T}(\mathfrak{X},\mathfrak{so}(2s+1),2r+1)$$

Proof. By Lemma 5.2, it is enough to show that the following equalities

$$\begin{aligned}&|\operatorname{Orb}_U|\prod_{q=1}^n\frac{\det\left(\zeta^{u_i(\lambda_q^j+r-j+\frac{1}{2})}-\zeta^{-u_i(\lambda_q^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{u_i(r-j+\frac{1}{2})}-\zeta^{-u_i(r-j+\frac{1}{2})}\right)}\left(\frac{\Phi_k(U)}{4k^r}\right)\\&=|\operatorname{Orb}_{U^c}|\prod_{q=1}^n\frac{\det\left(\zeta^{u_i^c((\lambda_q^T)^j+r-j+\frac{1}{2})}-\zeta^{-u_i^c((\lambda_q^T)^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{u_i^c(r-j+\frac{1}{2})}-\zeta^{-u_i^c(r-j+\frac{1}{2})}\right)}\left(\frac{\Phi_k(U^c)}{4k^s}\right),\end{aligned}$$

where $U\in P^+_{2s+1}(\mathfrak{so}(2r+1))/\Gamma$ and $r+s\in U$.

$$\begin{aligned}&|\operatorname{Orb}_{U'}|\prod_{q=1}^n\frac{\det\left(\zeta^{(u_i'-\frac{1}{2})(\lambda_q^j+r-j+\frac{1}{2})}-\zeta^{-(u_i'-\frac{1}{2})(\lambda_q^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{(u_i'-\frac{1}{2})(r-j+\frac{1}{2})}-\zeta^{-(u_i'-\frac{1}{2})(r-j+\frac{1}{2})}\right)}\left(\frac{\Phi_k(U'-\frac{1}{2})}{4k^r}\right)\\=\ &|\operatorname{Orb}_{((r+s+1)-U'^c)}|\prod_{q=1}^n\frac{\det\left(\zeta^{(u_i''-\frac{1}{2})((\lambda_q^T)^j+r-j+\frac{1}{2})}-\zeta^{-(u_i''-\frac{1}{2})((\lambda_q^T)^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{(u_i''-\frac{1}{2})(r-j+\frac{1}{2})}-\zeta^{-(u_i''-\frac{1}{2})(r-j+\frac{1}{2})}\right)}\left(\frac{\Phi_k((r+s+\frac{1}{2})-U'^c)}{4k^s}\right),\end{aligned}$$

where $U'\in P^0_{2s+1}(\mathfrak{so}(2r+1))/\Gamma$. Now by Lemma 8.4 and Lemma 8.5 we know that

$$|\operatorname{Orb}_U|\left(\frac{\Phi_k(U)}{4k^r}\right)=|\operatorname{Orb}_{U^c}|\left(\frac{\Phi_k(U^c)}{4k^s}\right).$$

$$\left(\frac{\Phi_k(U'-\frac{1}{2})}{4k^r}\right)=\left(\frac{\Phi_k((r+s+\frac{1}{2})-U'^c)}{4k^s}\right).$$

Thus we need to check to the following identity of determinants for the pair (U,U^c) which follows from Lemma 11.2.

$$\prod_{q=1}^{n}\frac{\det\left(\zeta^{u_i(\lambda_q^j+r-j+\frac{1}{2})}-\zeta^{-u_i(\lambda_q^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{u_i(r-j+\frac{1}{2})}-\zeta^{-u_i(r-j+\frac{1}{2})}\right)}=\prod_{q=1}^{n}\frac{\det\left(\zeta^{u_i^c((\lambda_q^T)^j+r-j+\frac{1}{2})}-\zeta^{-u_i^c((\lambda_q^T)^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{u_i^c(r-j+\frac{1}{2})}-\zeta^{-u_i^c(r-j+\frac{1}{2})}\right)}.$$

and

$$\prod_{q=1}^{n}\frac{\det\left(\zeta^{(u_i'-\frac{1}{2})(\lambda_q^j+r-j+\frac{1}{2})}-\zeta^{-(u_i'-\frac{1}{2})(\lambda_q^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{(u_i'-\frac{1}{2})(r-j+\frac{1}{2})}-\zeta^{-(u_i'-\frac{1}{2})(r-j+\frac{1}{2})}\right)}$$
$$=\prod_{q=1}^{n}\frac{\det\left(\zeta^{(u_i''-\frac{1}{2})((\lambda_q^T)^j+r-j+\frac{1}{2})}-\zeta^{-(u_i''-\frac{1}{2})((\lambda_q^T)^j+r-j+\frac{1}{2})}\right)}{\det\left(\zeta^{(u_i''-\frac{1}{2})(r-j+\frac{1}{2})}-\zeta^{-(u_i''-\frac{1}{2})(r-j+\frac{1}{2})}\right)},$$

for the pair (U',U'^c), where $\lambda_q^T=((\lambda_q^T)^1\geq\cdots\geq(\lambda_q^T)^s)$. This completes the proof. □

With the same notation and assumptions as the Proposition 8.6 we have the following Proposition.

Proposition 8.7. *If $\sum_{i=1}^n|\lambda|$ is even, then the following equality of dimensions holds.*

$$\dim_{\mathbb{C}}\mathcal{V}^{\dagger}_{\vec{\lambda}\cup\sigma(0)}(\mathfrak{X},\mathfrak{so}(2r+1),2r+1)=\dim_{\mathbb{C}}\mathcal{V}^{\dagger}_{\vec{\lambda}^T\cup\sigma(0)}(\mathfrak{X},\mathfrak{so}(2s+1),2r+1)$$

Proof. The proof follows from the proof of Proposition 8.6, Lemma 11.6 and Lemma 11.7 □

Let $\lambda_i\in\mathcal{Y}_{r,s}$ such that $\sum_{i=1}^n|\lambda_i|$ is odd and $\mathfrak{X}$ be the data associated to the n-distinct points on $\mathbb{P}^1$. Denote by $\vec{\lambda}$ the n-tuple of weights $(\lambda_1,\dots,\lambda_n)$ and $\vec{\lambda}^T$ the n-tuple of weights $(\lambda_1^T,\dots,\lambda_n^T)$. Consider the conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda}\cup 0}(\mathfrak{X},\mathfrak{so}(2r+1))$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}^T\cup\sigma(0)}(\mathfrak{X},\mathfrak{so}(2s+1))$.

Proposition 8.8.

$$\dim_{\mathbb{C}}\mathcal{V}^{\dagger}_{\vec{\lambda}\cup 0}(\mathfrak{X},\mathfrak{so}(2r+1),2s+1)=\dim_{\mathbb{C}}\mathcal{V}^{\dagger}_{\vec{\lambda}^T\cup\sigma(0)}(\mathfrak{X},\mathfrak{so}(2s+1),2r+1)$$

Remark 8.9. *These equalities of the dimensions of the conformal blocks gives rise to some interesting relations between the fusion ring of $\mathfrak{so}(2r+1)$ at level $2s+1$ with the fusion ring of $\mathfrak{so}(2s+1)$ at level $2r+1$. See [19] for more details.*

9. Highest Weight Vectors

In this section we briefly summarize the construction of the highest weight integrable modules $\mathcal{H}_0(\mathfrak{so}(2r+1))$ and $\mathcal{H}_{\omega_1}(\mathfrak{so}(2r+1))$. We use this to explicitly describe the highest weight vector of the components that appear in the branching. Our discussion follows closely the discussion in [13].

9.1. **Spin Modules.** We first recall the definition of the Clifford algebra. Let W be a vector space (not necessarily finite dimensional) with a non degenerate bilinear for $\{,\}$.

Definition 9.1. *We define the* Clifford Algebra *associated to W and $\{,\}$ to be*

$$C(W):=T(W)/I,$$

where $T(W)$ is the tensor algebra of W and I is the two sided ideal generated by elements of the form $v\otimes w+w\otimes v-\{v,w\}$.

9.1.1. *Spin Module of $C(W)$.* Suppose there exists an isotropic decomposition $W = W^+ \oplus W^-$, i.e. $\{W^{\pm}, W^{\pm}\} = 0$ and $\{,\}$ restricted to $W^+ \otimes W^-$ is non degenerate. Then the exterior algebra $\bigwedge W^-$ can be viewed as $\bigwedge W^-$-module by taking wedge product on the left. This gives rise to an irreducible $C(W)$-module structure on $\bigwedge W^-$ by defining

$$w^+.1 = 0,$$

for all $w^+ \in W^+$ and $1 \in \bigwedge W^-$.

Next if $W = W' \oplus \mathbb{C}e$ be an orthogonal direct sum with $\{e, e\} = 1$ and W' has an isotropic decomposition of the form $W^+ \oplus W^-$(we refer this as quasi isotropic decomposition of W). Then the $C(W')$-module $\bigwedge W^-$ described above becomes and irreducible $C(W)$ module by defining

$$\sqrt{2}e.v := \pm(-1)^p v \quad \text{for} \quad v \in \bigwedge^p W^-$$

Any element of W^-(respectively W^+) is called a creation operator (respectively annihilation operator).

9.1.2. *Root Spaces and basis of $\mathfrak{so}(2r+1)$.* Consider a finite dimensional vector space W_r of dimension $2r+1$ with a non-degenerate symmetric bilinear form $\{,\}$. Let $\{e_i\}_{i=-r}^{r}$ be an orthonormal basis of W_r. For $j > 0$ set

$$\phi^j = \frac{1}{\sqrt{2}}(e_j + \sqrt{-1}e_{-j}); \quad \phi^{-j} = \frac{1}{\sqrt{2}}(e_j - \sqrt{-1}e_{-j}) \ \text{ and } \ \phi^0 = e_0$$

Let $\phi^1, \dots, \phi^r, \phi^0, \phi^{-r}, \dots, \phi^{-1}$ be the chosen ordered basis of W_r. For any i, j we define $E^i_j(\phi^k) := \delta_{k,j}\phi^i$.

We identify the Lie algebra $\mathfrak{so}(2r+1)(W_r)$ with $\mathfrak{so}(2r+1)$ as follows

$$\mathfrak{so}(2r+1) := \{A \in \mathfrak{sl}(2r+1) | A^T J + JA = 0\},$$

where J is a $2r+1 \times 2r+1$ matrix

$$J = \begin{array}{cc} & \begin{array}{ccccccc} 1 & \dots & r & 0 & -r & \dots & -1 \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ r \\ 0 \\ -r \\ \vdots \\ -1 \end{array} & \begin{pmatrix} & & & & & & 1 \\ & 0 & & & & 1 & \\ & & & & 1 & & \\ & & & 1 & & & \\ & & 1 & & & & \\ & 1 & & & & 0 & \\ 1 & & & & & & \end{pmatrix} \end{array}$$

We put $B^i_j = E^i_j - E^{-j}_{-i}$ and we take the Cartan subalgebra $\mathfrak{h}$ to be the subalgebra of diagonal matrices. Clearly $\mathfrak{h} = \oplus_{j=1}^{r} \mathbb{C}B^j_j$. The corresponding dual basis of $\mathfrak{h}^*$ is L_j, where $L_j(B^k_k) = \delta_{j,k}$. The simple positive roots $\{\alpha_i\}_{i=1}^{r}$ of $\mathfrak{so}(2r+1)$ are given by $L_1 - L_2, \dots, L_{r-1} - L_r, L_r$. The root spaces of $\mathfrak{so}(2r+1)$ are of the form $\mathfrak{g}_{L_i \pm L_j} = \mathbb{C}B^i_{\mp j}$ and $\mathfrak{g}_{L_i} = \mathbb{C}B^i_0$.

Remark 9.2. *The basis of the vector space W_r chosen is different than that chosen* [10]*. Under the change of basis the description of the root spaces, roots in* [10] *matches up with the one described here(see* [13]*). In this section we prefer this basis because the branching formulas that we describe in the next section become simpler.*

9.1.3. *Spin module* $\bigwedge W_r^{\mathbb{Z}+\frac{1}{2},-}$ *of* $\widehat{\mathfrak{so}}(2r+1)$. Consider as before W_r to be a $2r+1$ dimensional vector with a non degenerate symmetric bilinear form $\{,\}$. Let $W_r^{\pm} = \oplus_{i=1}^{r}\mathbb{C}\phi^{\pm}$, then a quasi isotropic decomposition of W_r is given by

$$W_r = W_r^+ \oplus W_r^- \oplus \mathbb{C}\phi^0$$

We define a new vector space $W_r^{\mathbb{Z}+\frac{1}{2}}$ with an inner product $\{,\}$ as follows

$$W_r^{\mathbb{Z}+\frac{1}{2}} := W_r \otimes t^{\frac{1}{2}}\mathbb{C}[t,t^{-1}] \ \text{ with } \ \{w_1(a), w_2(b)\} = \{w_1, w_2\}\delta_{a+b,0},$$

where $w_1, w_2 \in W_r$; $a, b \in \mathbb{Z} + \frac{1}{2}$ and $w_1(a) = w_1 \otimes t^a$. We choose a quasi isotropic decomposition of $W_r^{\mathbb{Z}+\frac{1}{2}}$ is given as follows

$$W_r^{\mathbb{Z}+\frac{1}{2}} = W_r^{\mathbb{Z}+\frac{1}{2},+} \oplus W_r^{\mathbb{Z}+\frac{1}{2},-} \oplus \mathbb{C}\phi^0,$$

where $W_r^{\mathbb{Z}+\frac{1}{2},\pm} := W_r \otimes t^{\pm\frac{1}{2}}\mathbb{C}[t^{\pm 1}]$. We define the normal product ${}^o_o \quad {}^o_o$ for $w_1(a), w_2(b) \in W_r^{\mathbb{Z}+\frac{1}{2}}$ by the following:

$${}^o_o w_1(a)w_2(b){}^o_o = \begin{cases} -w_2(b)w_1(a) & \text{if } a > 0 > b, \\ \frac{1}{2}(w_1(a)w_2(b) - w_2(b)w_1(a)) & \text{if } a = b = 0, \\ w_1(a)w_2(b) & \text{otherwise} \end{cases}$$

We now describe the action of $\widehat{\mathfrak{so}}(2r+1)$ on $\bigwedge W_r^{\mathbb{Z}+\frac{1}{2},-}$ and the explicit description of the level 1 $\widehat{\mathfrak{so}}(2r+1)$ modules $\mathcal{H}_0(\mathfrak{so}(2r+1))$ and $\mathcal{H}_{\omega_1}(\mathfrak{so}(2r+1))$. For a proof we refer the reader to [7].

Proposition 9.3. *The following map is a Lie algebra monomorphism*

$$\widehat{\mathfrak{so}}(2r+1) \to \mathrm{End}(\bigwedge W_r^{\mathbb{Z}+\frac{1}{2},-}),$$

$$B^i_j(m) \to \sum_{a+b=m} {}^0_0\phi^i(a)\phi^{-j}(b){}^0_0; \quad c \to \mathrm{id}\,.$$

Proposition 9.4. *Suppose* $r \geq 1$, *then the following are isomorphic as* $\widehat{\mathfrak{so}}(2r+1)$ *modules:*

(1) $\bigwedge^{even}(W_r^{\mathbb{Z}+\frac{1}{2},-}) \simeq \mathcal{H}_0(\mathfrak{so}(2r+1))$.
(2) $\bigwedge^{odd}(W_r^{\mathbb{Z}+\frac{1}{2},-}) \simeq \mathcal{H}_{\omega_1}(\mathfrak{so}(2r+1)$.

The highest weight vectors is given by 1, $\phi^1(-\frac{1}{2}).1$ *respectively.*

9.2. **Highest Weight Vectors.** Let W_s be a $2s+1$ dimensional vector space over $\mathbb{C}$ with a non degenerate bilinear form $\{,\}$ and $\{e_p\}_{p=1}^s$ be a orthonormal basis of W_s and let $\phi^1, \ldots, \phi^s, \phi^0, \phi^{-s}, \ldots, \phi^{-1}$ be an ordered isotropic basis of W_s as before. The tensor product of $W_d = W_r \otimes W_s$ carries a non degenerate symmetric bilinear form $\{,\}$ given by the product of the forms on W_r and W_s. Clearly for $-r \leq j, \leq r; -s \leq p \leq s$ the elements $\{e_{j,p} := e_j \otimes e_p\}$ form an orthonormal basis of W_d. By $(j,p) > 0$ we mean $j > 0$ or $j = 0, p > 0$ and put

$$\phi^{j,p} = \frac{1}{\sqrt{2}}(e_{j,p} - \sqrt{-1}e_{-j,-p}); \quad \phi^{-j,-p} = \frac{1}{\sqrt{2}}(e_{j,p} + \sqrt{-1}e_{-j,-p})$$

for $(j,p) > 0$. The form $\{,\}$ on W_d is given by the formula

$$\{\phi^{j,p}, \phi^{-k,-q}\} = \delta_{j,k}\delta_{p,q}, \quad \text{for} \quad -r \leq j,k \leq r; -s \leq p,q \leq s$$

Let as before $W_d^{\pm} = \bigoplus_{(j,p)>0} \mathbb{C}\phi^{\pm j,\pm p}$ and $\phi^{0,0} = e_{0,0}$. The quasi isotropic decomposition of W_N is given as follows

$$W_d = W_d^+ \oplus W_d^- \oplus \mathbb{C}\phi^{0,0}$$

We define the operator $E_{k,q}^{j,p}$ by $E_{k,q}^{j,p}(\phi^{i,l}) = \delta_{i,k}\delta_{l,q}\phi^{j,p}$ and $B_{k,q}^{j,p} = E_{k,q}^{j,p} - E_{-j,-p}^{-k,-q}$ and the Cartan subalgebra $\mathfrak{H}$ to be subalgebra generated by the diagonal matrices. Clearly $\mathfrak{H} = \oplus_{(j,p)>0}\mathbb{C}B_{j,p}^{j,p}$. Let $L_{j,p}$ be the dual of $B_{j,p}^{j,p}$ for $(j,p) > 0$. Thus $\mathfrak{H}^* = \oplus_{(j,p)>0}\mathbb{C}L_{j,p}$

9.2.1. *Highest Weight Vectors as Wedge product.* To every Young diagram in $\mathcal{Y}_{r,s}$ we associate an $2r+1 \times 2r+1$ matrix. First to every Young diagram λ we associate a $r \times s$ matrix $Y(\lambda)$ as follows

$$Y(\lambda)_{i,j} = \begin{cases} 0 & \text{if } \lambda \text{ has a box in the } (i,j) \text{ position} \\ 1 & \text{otherwise} \end{cases}$$

Finally to $Y(\lambda)$ we associate the matrix

$$\widetilde{Y}(\lambda) = \begin{array}{c} \\ 1 \\ \vdots \\ r \\ 0 \\ -r \\ \vdots \\ -1 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & \dots & s & 0 & -s & \dots & -1 \end{array} \\ \left(\begin{array}{ccccccc} & & & 1 & 1 & \dots & 1 \\ & Y(\lambda) & & \vdots & \vdots & \dots & \vdots \\ & & & 1 & 1 & \dots & 1 \\ 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right) \end{array}$$

For $\lambda \in \mathcal{Y}_{r,s}$ let $\widetilde{Y}(\lambda)$ be the image of $\mathcal{Y}_{r,s}$ we define the following operations

$$\sigma^L(\widetilde{Y}(\lambda))_{j,p} := \widetilde{Y}(\lambda)_{j,p} - \delta_{j,1}\delta_{\widetilde{Y}_{1,|p|},1} \tag{9.1}$$

$$\sigma^R(\widetilde{Y}(\lambda))_{j,p} := \widetilde{Y}(\lambda)_{j,p} - \delta_{p,1}\delta_{\widetilde{Y}_{|j|,1},1} \tag{9.2}$$

The following Proposition in [13] gives the highest weight vectors for the branching rule described in Section 5.3

Proposition 9.5. *The vector* $\big(\bigwedge_{\widetilde{y}_{j,p}=0} \phi^{j,p}(-\frac{1}{2})\big).1$ *defined up to a sign for each of the matrices* $\widetilde{Y}(\lambda)$, $\sigma^L(\widetilde{Y}(\lambda))$, $\sigma^R(\widetilde{Y}(\lambda))$, $\sigma^L(\sigma^R(\widetilde{Y}(\lambda)))$ *gives a highest weight vector of the components with highest weight* (λ, λ^T), $(\sigma(\lambda), \lambda^T)$, $(\lambda, \sigma(\lambda^T))$ *and* $(\sigma(\lambda), \sigma(\lambda^T))$

Next we describe the highest vectors for some of the components in the "Kac-Moody" form. We use this explicit descriptions to prove the basic cases of the rank-level duality conjectures.

9.2.2. *Highest weight vectors in Kac-Moody form.* Let $\lambda, \lambda' \in \mathcal{Y}_{r,s}$ and assume that λ is obtained from $\lambda' \in \mathcal{Y}_{r,s}$ by adding two boxes. In terms of the matrices described in the previous section $Y(\lambda)$ is obtained from $Y(\lambda')$ by making the 1 to 0 in exactly two places of $Y(\lambda')$ say at (a,b) and (c,d). Assume that $(a,b)<(c,d)$ under the Lexicographic ordering.

The following Proposition describes the highest weight vectors in the "Kac-Moody" form i.e. as elements of universal enveloping of $\widehat{\mathfrak{so}}(2d+1)$ acting on the highest weight vectors of $\mathcal{H}_0(\mathfrak{so}(2d+1))$ and $\mathcal{H}_{\omega_1}(\mathfrak{so}(2d+1))$.

Proposition 9.6. *Let λ and λ' be as before. Then the following holds:*

(1) *If $v_{\lambda'} \in \operatorname{End}(\bigwedge W_d^{\mathbb{Z}+\frac{1}{2},-})$ be the highest weight vector of the component $\mathcal{H}_{\lambda'}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda'^T}(\mathfrak{so}(2s+1))$, then the highest weight vector v_λ of the component $\mathcal{H}_\lambda(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$ is given by the following:*

$$v_\lambda = B^{a,b}_{-c,-d}(-1).v_{\lambda'}.$$

(2) *If $v^{\lambda'}$ is the lowest weight vector of $\mathcal{H}_{\lambda'}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda'^T}(\mathfrak{so}(2s+1))$, then the lowest weight vector v^λ of the component $\mathcal{H}_\lambda(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\lambda^T}(\mathfrak{so}(2s+1))$ is given by the following:*

$$v^\lambda = B^{-a,-b}_{c,d}(-1).v^{\lambda'}.$$

Proof. The proof of the above easily follows from Proposition 9.5 and Proposition 9.3. □

Remark 9.7. *There is no unique way of building a Young diagram λ starting from the empty Young diagram. So there is no uniqueness in the expressions of the highest weight vectors described in Proposition 9.6.*

10. Proof of Theorem 7.1

In this section we give a proof of Theorem 7.1. The main steps of the proof are summarized below.

10.1. **Key Steps.** The strategy of the proof of Theorem 7.1 follows closely to [1] and [16] but has some significant differences in the individual steps.

10.1.1. *Step I.* We study the degeneration of the rank-level duality map on $\mathbb{P}^1$ with n marked points. We use Proposition 10.10 to reduce to case when the conformal blocks is on $\mathbb{P}^1$ with 3 marked points and the representation attached to one of the marked points in ω_1. The details of this step are explained in Section 10.4.

10.1.2. *Step II.* We are now reduced to proving the rank-level duality for any any admissible pair of the form $(\omega_1, \lambda_2, \lambda_3)$ and $(\omega_1, \beta_1, \beta_2)$. We use Proposition 6.4 to determine which conformal blocks on $\mathbb{P}^1$ with 3 marked points with representations of the form $(\omega_1, \lambda_2, \lambda_3)$ are non zero.

10.1.3. *Step III.* We use Proposition 3.6 and further reduce to proving the rank-duality for 3-pointed curves for admissible pairs of the following forms:

(1) $(\omega_1, \lambda_2, \lambda_3); (\omega_1, \lambda_2^T, \lambda_3^T)$, where $\lambda_2, \lambda_3 \in \mathcal{Y}_{r,s}$ and λ_2 is obtained by λ_3 either by adding or deleting a box. The rank-level duality for these cases are proved in Section 10.2.5 and Section 10.2.
(2) $(\omega_1, \lambda, \lambda); (\omega_1, \lambda^T, \sigma(\lambda^T))$ where $\lambda \in \mathcal{Y}_{r,s}$ and $(\lambda, L_r) \neq 0$. The rank-level duality for the above these cases are proved in Section 10.3.

10.2. **The minimal 3-point cases.** In this section, we prove the rank-level duality isomorphism for 1 dimensional conformal blocks on $\mathbb{P}^1$ with 3 marked points. We use these cases to prove the rank-level duality isomorphism in the general case. Through out this section, we will assume that $(P_1, P_2, P_3) = (1, 0, \infty)$ with coordinates $\xi_1 = z - 1$, $\xi_2 = z$ and $\xi_3 = \frac{1}{z}$. We denote by $\mathfrak{X}$ the associated data. Let $\lambda_2, \lambda_3 \in \mathcal{Y}_{r,s}$, $\vec{\lambda} = (\omega_1, \lambda_2, \lambda_3)$, $\vec{\Lambda} = (\omega_1, \omega_1, 0)$ and λ_2 is obtained from λ_3 is adding or deleting a box.

Remark 10.1. *The following strategy is influenced by the proof of Proposition 6.3 in* [1].

Let us summarize our main steps to prove these minimal cases.

10.2.1. *Step I.* We always choose $|\Phi_2\rangle$(respectively $|\Phi_3\rangle$) to be the highest (respectively lowest) weight vector of the module with highest weight (λ_2, λ_2^T) (respectively (λ_3, λ_3^T)).

10.2.2. *Step II.* If λ_3 is obtained from λ_2 by adding a box in the (a, b)-th coordinate, then we choose $|\Phi_1\rangle$ to be $\phi^{a,b}(-\frac{1}{2})$. If λ_2 is obtained from λ_3 by adding a box in the (a, b)-th coordinate, then we choose $|\Phi_1\rangle$ to be $\phi^{-a,-b}(-\frac{1}{2})$. With this choice it is clear that the $\mathfrak{H}$ weights of $|\Phi_1 \otimes \Phi_2 \otimes \Phi_3\rangle$ is zero.

10.2.3. *Step III.* We use induction on $\max(|\lambda_2|, |\lambda_3|)$. The basic cases of induction are proved in Section 10.2.5. Assume that $|\lambda_2| = |\lambda_3| + 1$. Let $\lambda_2' \in \mathcal{Y}_{r,s}$ be such that

(1) λ_2 is obtained by adding two boxes from λ_2'.
(2) λ_3 is obtained by adding a box to λ_2'. (The other case $|\lambda_3| = |\lambda_2| + 1$ is handled similarly)

10.2.4. *Step IV.* We use gauge symmetry to reduce to the case $(\omega_1, \lambda_2', \lambda_3)$. This is done in Proposition 10.6. Now $\max(|\lambda_2'|, |\lambda_3|) < |\lambda_2|$. (The other case is handled similarly). Hence we are done by induction.

Remark 10.2. *The minimal cases here are similar to the minimal cases in* [1]. *In the case of symplectic rank-level duality, T. Abe identified the rank-level duality map with the symplectic strange duality map and used the geometry of parabolic bundles with a symplectic form to prove that the rank-level duality maps are non zero. As remarked earlier, we were not able to describe the map in Theorem 7.1 geometrically. However the steps described above can be used to tackle minimal cases in* [1].

Now we focus on the proof of the rank-level duality for the one dimensional conformal blocks discussed in the strategy.

10.2.5. *The basic cases for induction.* The finite dimensional irreducible $\mathfrak{so}(2d+1)$ module V_{ω_1} can be realized inside $\bigwedge^{odd} W_d^{\mathbb{Z}+\frac{1}{2},-}$ as vectors of the form $\phi^{i,j}(-\frac{1}{2})$. On V_{ω_1} there is a canonical $\mathfrak{so}(2d+1)$ invariant bilinear form Q given by the following formula.

$$Q(\phi^{j,p}(-\frac{1}{2}), \phi^{-k,-q}(-\frac{1}{2})) = \delta_{j,k}\delta_{p,q}$$

We think of $\mathbb{P}^1$ as $\mathbb{C}\cup\{\infty\}$ and z be a global coordinate of $\mathbb{C}$. Through this section we assume that the points $P_1 = 1$ with coordinate $\xi_1 = z-1$, $P_2 = 0$ with coordinate $\xi_2 = z$ and the point $P_3 = \infty$ with coordinate $\xi_3 = \frac{1}{z}$ and $\mathfrak{X}$ be the associated to $\mathbb{P}^1$ and points $\vec{p} = (P_1, P_2, P_3)$ with chosen coordinates. Also let $\vec{\Lambda} = (\omega_1, \omega_1, 0)$.

Lemma 10.3. *Let $\vec{\lambda} = (\omega_1, \omega_1, 0)$. Then the following rank-level duality map*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \otimes \mathcal{V}_{\vec{\lambda^T}}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1) \to \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1)$$

is non zero.

Proof. Let $\langle\Psi'| \in \mathcal{V}^{\dagger}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1)$ be a non zero element. It is enough to produce $|\Phi_1 \otimes \Phi_2 \otimes \Phi_3\rangle \in \mathcal{H}_{\vec{\lambda}}(\mathfrak{so}(2r+1) \otimes \mathcal{H}_{\vec{\lambda^T}}(\mathfrak{so}(2s+1))$ such that

$$\langle\Psi'|\Phi_1 \otimes \Phi_2 \otimes \Phi_3\rangle \neq 0.$$

We choose $|\Phi_1\rangle = \phi^{-1,-1}(-\frac{1}{2})$, $|\Phi_2\rangle = \phi^{-1,-1}(-\frac{1}{2})$ and $|\Phi_3\rangle = 1$. By Propagation of Vacua we get

$$\begin{aligned}\langle\Psi'|\phi^{-1,-1}(-\frac{1}{2}) \otimes \phi^{1,1}(-\frac{1}{2}) \otimes 1\rangle &= \langle\Psi|\phi^{-1,-1}(-\frac{1}{2}) \otimes \phi^{1,1}(-\frac{1}{2})\rangle, \\ &= Q(\phi^{-1,-1}(-\frac{1}{2}), \phi^{1,1}(-\frac{1}{2})), \\ &= 1.\end{aligned}$$

□

Lemma 10.4. *Let $\vec{\lambda} = (\omega_1, \omega_1, 2\omega_1)$ or $(\omega_1, \omega_1, \omega_2)$. Then the following rank-level duality map*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \otimes \mathcal{V}_{\vec{\lambda^T}}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1) \to \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1),$$

is non zero.

Proof. First let $\vec{\lambda} = (\omega_1, \omega_1, 2\omega_1)$. We choose $|\Phi_3\rangle$ to be the lowest weight vector of the module $\mathcal{H}_{2\omega_1}(\mathfrak{so}(2r+1)) \otimes \mathcal{H}_{\omega_2}(\mathfrak{so}(2s+1))$, $|\Phi_2\rangle = \phi^{1,1}(-\frac{1}{2})$. We choose $|\Phi_1\rangle$ such that the $\mathfrak{H}$ weight of $|\Phi_1 \otimes \Phi_2 \otimes \Phi_3\rangle$ is zero. In this case $|\Phi_1\rangle = \phi^{1,2}(-\frac{1}{2})$. By "gauge symmetry", we get the

following:

$$
\begin{aligned}
&\langle \Psi' | \phi^{1,2}(-\frac{1}{2}) \otimes \phi^{1,1}(-\frac{1}{2}) \otimes B^{-1,-1}_{1,2}(-1).1\rangle \\
&= \langle \Psi' | \phi^{1,2}(-\frac{1}{2}) \otimes \phi^{1,1}(-\frac{1}{2}) \otimes B^{-1,-1}_{1,2}(\frac{1}{\xi_3}).1\rangle, \\
&= -\langle \Psi' | B^{-1,-1}_{1,2}\phi^{1,2}(-\frac{1}{2}) \otimes \phi^{1,1}(-\frac{1}{2}) \otimes 1\rangle - \langle \Psi' | \phi^{1,2}(-\frac{1}{2}) \otimes B^{-1,-1}_{1,2}(1).\phi^{1,1}(-\frac{1}{2}) \otimes 1\rangle, \\
&= -\langle \Psi' | \phi^{-1,-1}(-\frac{1}{2}) \otimes \phi^{1,1}(-\frac{1}{2}) \otimes 1\rangle \ \ [\text{Since } B^{-1,-1}_{1,2}(1).\phi^{1,1}(-\tfrac{1}{2}) = 0], \\
&\neq 0. \ \ [\text{By Proposition 10.3}]
\end{aligned}
$$

The case $\vec{\lambda} = (\omega_1, \omega_1, \omega_2)$ follows similarly.

□

Lemma 10.5. *Let $\vec{\lambda} = (\omega_1, \omega_1 + \omega_2, 2\omega_1)$ or $(\omega_1, \omega_1 + \omega_2, \omega_2)$. Then the following rank-level duality map:*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \otimes \mathcal{V}_{\vec{\lambda}^T}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1) \to \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1),$$

is non zero.

Proof. Let $\lambda_2' = \omega_1$ and $\lambda_2 = \omega_1 + \omega_2$. Since λ_2 is obtained from λ_2' by adding two boxes in the $(1,2)$ and $(2,1)$ coordinate. Thus by Proposition 9.6, we get

$$v_{\lambda_2} = B^{1,2}_{-2,-1}(-1).\phi^{1,1}(-\frac{1}{2}).$$

As in Proposition 10.4 the vector $|\Phi_3\rangle = B^{-1,-1}_{1,2}(-1).1$. We choose $|\Phi_2\rangle = v_{\lambda_2}$ and $|\Phi_1\rangle$ such that the $\mathfrak{H}$ weight of $|\Phi_1 \otimes \Phi_2 \otimes \Phi_2\rangle$ is zero. In this case $|\Phi_1\rangle = \phi^{-2,-1}(-\frac{1}{2})$. By gauge symmetry, we get the following:

$$
\begin{aligned}
&\langle \Psi' | \phi^{-2,-1}(-\frac{1}{2}) \otimes B^{1,2}_{-2,-1}(-1)\phi^{1,1}(-\frac{1}{2}) \otimes B^{-1,-1}_{1,2}(-1).1\rangle \\
&= -\langle \Psi' | B^{1,2}_{-2,-1}\phi^{-2,-1}(-\frac{1}{2}) \otimes \phi^{1,1}(-\frac{1}{2}) \otimes B^{-1,-1}_{1,2}(-1).1\rangle \\
&\quad -\langle \Psi' | \phi^{-2,-1}(-\frac{1}{2}) \otimes \phi^{1,1}(-\frac{1}{2}) \otimes B^{1,2}_{-2,-1}(1)B^{-1,-1}_{1,2}(-1).1\rangle, \\
&= -\langle \Psi' | B^{1,2}_{-2,-1}\phi^{-2,-1}(-\frac{1}{2}) \otimes \phi^{1,1}(-\frac{1}{2}) \otimes B^{-1,-1}_{1,2}(-1).1\rangle \\
&\quad -\langle \Psi' | \phi^{-2,-1}(-\frac{1}{2}) \otimes \phi^{1,1}(-\frac{1}{2}) \otimes B^{-1,-1}_{1,2}(-1)B^{1,2}_{-2,-1}(1).1\rangle \\
&\quad -\langle \Psi' | \phi^{-2,-1}(-\frac{1}{2}) \otimes \phi^{1,1}(-\frac{1}{2}) \otimes [B^{1,2}_{-2,-1}(1), B^{-1,-1}_{1,2}(-1)].1\rangle, \\
&= -\langle \Psi' | \phi^{1,2}(-\frac{1}{2}) \otimes \phi^{1,1}(-\frac{1}{2}) \otimes B^{-1,-1}_{1,2}(-1).1\rangle, \\
&\neq 0. \ \ [\text{By Proposition 10.4}]
\end{aligned}
$$

□

10.2.6. *The inductive step.*

Proposition 10.6. *Let $|\lambda_2| = |\lambda_3| + 1$ and λ_2 is obtained from λ_3 by adding a box in the (c,d)-th coordinate. Further assume that λ_3 is obtained from λ_2' by adding a box in the (a,b)-th coordinate. Then the following are equivalent:*

1. *The rank-level duality map for the pair $(\omega_1, \lambda_2, \lambda_3); (\omega_1, \lambda_2^T, \lambda_3^T)$ is non zero.*
2. *The rank-level duality map for the pair $(\omega_1, \lambda_2', \lambda_3); (\omega_1, \lambda_2'^T, \lambda_3^T)$ is non zero.*

Proof. Without loss of generality assume that $(a,b) < (c,d)$. Let $\langle \Psi'| \in \mathcal{V}^{\dagger}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1)$ be non zero. We choose $|\Phi_1\rangle = \phi^{-a,-b}(-\frac{1}{2})$, $|\Phi_2\rangle = B^{a,b}_{-c,-d}(-1)v_{\lambda_2'}$ and $|\Phi_3\rangle$ to be the lowest weight vector of the component. Then we have the following:

$$\begin{aligned}
&\langle \Psi' | \Phi_1 \otimes \Phi_2 \otimes \Phi_3 \rangle \\
&= \langle \Psi' | \phi^{-a,-b}(-\frac{1}{2}) \otimes B^{a,b}_{-c,-d}(-1)v_{\lambda_2'} \otimes \Phi_3 \rangle, \\
&= -\langle \Psi' | B^{a,b}_{-c,-d}\phi^{-a,-b}(-\frac{1}{2}) \otimes v_{\lambda_2'} \otimes \Phi_3 \rangle \\
&\qquad -\langle \Psi' | \phi^{-a,-b}(-\frac{1}{2}) \otimes v_{\lambda_2'} \otimes B^{a,b}_{-c,-d}(1)\Phi_3 \rangle, \\
&= \langle \Psi' | \phi^{c,d}(-\frac{1}{2}) \otimes v_{\lambda_2'} \otimes \Phi_3 \rangle. \quad (\text{ By Lemma 10.7})
\end{aligned}$$

The last expression is exactly the one that one consider to prove the rank-level duality for the pair $(\omega_1, \lambda_2', \lambda_3); (\omega_1, \lambda_2'^T, \lambda_3^T)$. Hence we are done. □

Lemma 10.7. *With the above notation we have the following*

$$B^{a,b}_{-c,-d}(1)|\Phi_3\rangle = 0.$$

Proof. Since $|\lambda_3|$ is even, the lowest weight vector $|\Phi_3\rangle$ is of the form $B^{-a,-b}_{e,f}(-1)v$. Moreover v has the form $\prod_{\alpha \in I} X_{-\alpha}(-1).1$ such that $(L_{a,b}, \alpha) = 0$ where I is a subset of positive root of $\mathfrak{so}(2d+1)$ and $X_{-\alpha}$ is a non-zero element in the weight space of the $-\alpha$.

$$\begin{aligned}
B^{a,b}_{-c,-d}(1)|\Phi_3\rangle &= B^{a,b}_{-c,-d}(1)B^{-a,-b}_{e,f}(-1)v, \\
&= B^{-a,-b}_{c,d}(-1)B^{a,b}_{-c,-d}(1)v + [B^{a,b}_{c,d}(1), B^{-a,-b}_{e,f}(-1)]v, \\
&= B^{-a,-b}_{c,d}(-1)B^{a,b}_{-c,-d}(1)\prod_{\alpha \in I} X_{-\alpha}(-).1 + [B^{a,b}_{c,d}, B^{-a,-b}_{e,f}]\prod_{\alpha \in I} X_{-\alpha}(-1).1, \\
&= B^{-a,-b}_{c,d}(-1)\big(\prod_{\alpha \in I} X_{-\alpha}(-1)\big)B^{a,b}_{-c,-d}(1).1 + \big(\prod_{\alpha \in I} X_{-\alpha}(-1)\big)[B^{a,b}_{c,d}, B^{-a,-b}_{e,f}].1, \\
&= 0.
\end{aligned}$$

Hence the Lemma follows.

□

The following Proposition has a similar proof to the previous one and tackles the case $|\lambda_3| = |\lambda_2| + 1$.

Proposition 10.8. *Let $|\lambda_3| = |\lambda_2| + 1$ and λ_3 is obtained from λ_2 by adding a box in the (c,d)-th coordinate. Further assume that λ_2 is obtained from λ_3' by adding a box in the (a,b)-th coordinate. Then the following are equivalent:*

(1) *The rank-level duality map for the pair $(\omega_1, \lambda_2, \lambda_3); (\omega_1, \lambda_2^T, \lambda_3^T)$ is non zero.*
(2) *The rank-level duality map for the pair $(\omega_1, \lambda_2, \lambda_3'); (\omega_1, \lambda_2^T, \lambda_3'^T)$ is non zero.*

10.3. **The remaining 3-point cases.** As before, we will assume that $(P_1, P_2, P_3) = (1, 0, \infty)$ with coordinates $\xi_1 = z - 1$, $\xi_2 = z$ and $\xi_3 = \frac{1}{z}$. We denote by $\mathfrak{X}$ the associated data. Let $\vec{\lambda} = (\omega_1, \lambda, \lambda)$, $\vec{\Lambda} = (\omega_1, \omega_1, 0)$, where $\lambda \in \mathcal{Y}_{r,s}$ such that $(\lambda, L_r) \neq 0$. The proof of the next Proposition follows the same pattern as Proposition 10.6. We give a proof of the first part for completeness.

Proposition 10.9. *The following rank-level duality maps are non-zero*

(1) *Let $|\lambda|$ is odd and $\vec{\lambda^T} = (\omega_1, \lambda^T, \sigma(\lambda^T))$*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \otimes \mathcal{V}_{\vec{\lambda^T}}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1) \to \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1)$$

(2) *Let $|\lambda|$ is even and $\vec{\lambda^T} = (\omega_1, \sigma(\lambda^T), \lambda^T)$*

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \otimes \mathcal{V}_{\vec{\lambda^T}}(\mathfrak{X}, \mathfrak{so}(2s+1), 2r+1) \to \mathcal{V}_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{so}(2d+1), 1)$$

Proof. Let $\lambda' \in \mathcal{Y}_{r,s}$ be such that $\sigma(\lambda)$ is obtained by adding boxes in $(0,1)$ and (r,a) to λ' and λ is obtained by adding a box in the (r,a)-th position. Since $|\lambda|$ is odd, the module with highest weight $(\lambda, \sigma(\lambda^T))$ appears in the branching of $\mathcal{H}_0(\mathfrak{so}(2d+1))$. By Proposition 9.5, the lowest weight vector is given by $B^{0,-1}_{r,a}(-1)v^{\lambda'}$, where $v^{\lambda'}$ is the lowest weight vector of the irreducible module with highest weight (λ', λ'^T).

As before we choose $|\Phi_3\rangle$ to be the lowest weight vector of the module with highest weight $(\lambda, \sigma(\lambda^T))$ and $|\Phi_2\rangle$ to be the highest weight vector v_λ and $|\Phi_1\rangle$ to be such that the $\mathfrak{H}$ weight of $|\Phi_1 \otimes \Phi_2 \otimes \Phi_3\rangle$ is zero. In this case $|\Phi_1\rangle$ is $\phi^{0,1}(-\frac{1}{2})$.

Let $\langle\Psi'| \in \mathcal{V}^\dagger_{\vec{\Lambda}}(\mathfrak{X}, \mathfrak{g}, 1)$ be a nonzero element. We use gauge symmetry as before to get the following

$$\begin{aligned}
&\langle\Psi'|\Phi_1 \otimes \Phi_2 \otimes \Phi_3\rangle \\
&= \langle\Psi'|\phi^{0,1}(-\frac{1}{2}) \otimes v_\lambda \otimes B^{0,-1}_{r,a}(-1)v^{\lambda'}\rangle, \\
&= -\langle\Psi'|B^{0,-1}_{r,a}(-1)\phi^{0,1}(-\frac{1}{2}) \otimes v_\lambda \otimes v^{\lambda'}\rangle \\
&\quad -\langle\Psi'|\phi^{0,1}(-\frac{1}{2}) \otimes B^{0,-1}_{r,a}(1)v_\lambda \otimes v^{\lambda'}\rangle, \\
&= \langle\Psi'|\phi^{-r,-a}(-\frac{1}{2}) \otimes v_\lambda \otimes v^{\lambda'}\rangle. \quad \text{(By Lemma similar to 10.7)}
\end{aligned}$$

Now the we known that $\langle\Psi'|\phi^{-r,-a}(-\frac{1}{2}) \otimes v_\lambda \otimes v^{\lambda'}\rangle \neq 0$ since rank-level duality holds for the pair $(\omega_1, \lambda, \lambda'); (\omega_1, \lambda^T, \lambda'^T)$.

□

10.4. **The proof in the general case.** In this section we give a proof of Theorem 7.1. First we formulate and prove a key degeneration result using the compatibility of rank-level duality and factorization discussed earlier. Let $\vec{\lambda}_1, \vec{\lambda}_2$ be n_1, n_2 tuples of weights in $P^0_{2s+1}(\mathfrak{so}(2r+1))$. Consider an n-tuple $\vec{\lambda} = (\vec{\lambda}_1, \vec{\lambda}_2)$ of weights in $P^0_{2s+1}(\mathfrak{so}(2r+1))$. Similarly consider $\vec{\mu} = (\vec{\mu}_1, \vec{\mu}_2)$ a (n_1+n_2) tuple of weights in $P^0_{2r+1}(\mathfrak{so}(2r+1))$ such that $\vec{\lambda}, \vec{\mu}$ is an admissible pair.

Proposition 10.10. *With the above notation, the following statements are equivalent:*

(1) *The rank-level duality map for the admissible pair $\vec{\lambda}, \vec{\mu}$ is an isomorphism for conformal blocks on $\mathbb{P}^1$ with n-marked points.*
(2) *The following rank-level duality maps for the admissible pairs are all isomorphic.*
   - *The rank-level duality maps are isomorphisms for all admissible pairs of the form $\vec{\lambda_1} \cup \lambda, \vec{\mu_1} \cup \mu$ for conformal blocks on $\mathbb{P}^1$ with (n_1+1) marked points.*
   - *The rank-level duality maps are isomorphisms for all the admissible pairs of the form $\vec{\lambda_2} \cup \lambda, \vec{\mu_2} \cup \mu$ for conformal blocks on $\mathbb{P}^1$ with (n_2+1) marked points.*

Proof. The proof follows from Corollary 4.5 and Proposition 3.3. Details will be added later. □

An immediate corollary of the Proposition 10.10 is the following:

Corollary 10.11. *If the rank-level duality conjecture holds for $\mathbb{P}^1$ with 3 marked points then it holds true for $\mathbb{P}^1$ with arbitrary number of marked points .*

By Proposition 3.6, we can further reduce to prove the rank level duality for a admissible pair of the form $(\lambda_1, \lambda_2, \lambda)$ and $(\lambda_1^T, \lambda_2^T, \beta)$ where $\lambda_1, \lambda_2 \in \mathcal{Y}_{r,s}$. Let $\vec{\lambda} = (\omega_1, \dots, \omega_1, \lambda, \lambda_2)$ and $\vec{\mu} = (\omega_1, \dots, \omega_1, \beta, \lambda_2^T)$, where $\lambda_1, \lambda_2 \in \mathcal{Y}_{r,s}$ and the number of ω_1's are $|\lambda_1|$. Clearly the pair $\vec{\lambda}$ and $\vec{\mu}$ are admissible. The following corollary follows directly from Proposition 10.10 and Lemma 10.13.

Corollary 10.12. *Let $\lambda_1, \lambda_2 \in \mathcal{Y}_{r,s}$. If the rank-level duality is an isomorphism for any $\mathbb{P}^1$ with $|\lambda_1| + 2$ marked points for the admissible pair $\vec{\lambda} = (\omega_1, \dots, \omega_1, \lambda, \lambda_2)$ and $\vec{\mu} = (\omega_1, \dots, \beta, \lambda_2^T)$, then the rank-level duality on $\mathbb{P}^1$ with 3 marked points is also an isomorphism for the admissible pairs $(\lambda_1, \lambda, \lambda_2)$ and $(\lambda_1^T, \beta, \lambda_2^T)$.*

Lemma 10.13. *Let $\lambda \in \mathcal{Y}_{r,s}$, and $\vec{\lambda} = (\lambda, \omega_1, \dots, \omega_1)$ where the number of ω_1 is $|\lambda|$, then*

$$\dim_{\mathbb{C}} \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}, \mathfrak{so}(2r+1), 2s+1) \neq 0$$

Proof. The proof follows directly by factorization of fusion coefficients and induction on $|\lambda|$. □

10.4.1. *Reduction to the one dimensional cases.* In the previous section we reduced Theorem 7.1 to the for admissible pairs of the form $\vec{\lambda} = (\omega_1, \dots, \omega_1, \lambda, \lambda_2)$ and $\vec{\mu} = (\omega_1, \dots, \omega_1, \beta, \lambda_2^T)$ where $\lambda \in P^0_{2s+1}(\mathfrak{so}(2r+1))$, the number of ω_1's are $|\lambda|$, $\lambda_2 \in \mathcal{Y}_{r,s}$ and $\beta \in P^0_{2r+1}(\mathfrak{so}(2s+1))$. The following Lemma shows that we can further reduce to the case for certain 1 dimensional conformal block on $\mathbb{P}^1$ with 3 marked points.

Lemma 10.14. *Let $\lambda_1, \lambda_2 \in P^0_{2s+1}(\mathfrak{so}(2r+1))$ and $\beta_1, \beta_2 \in P^0_{2r+1}(\mathfrak{so}(2s+1))$. If the rank-level duality holds for any admissible pairs of the form $(\lambda_1, \omega_1, \lambda_2)$ and $(\beta_1, \omega_1, \beta_2)$, then the rank-level duality holds for any admissible pairs on $\mathbb{P}^1$ with arbitrary number of marked points.*

Proof. The proof follows directly from Proposition 10.10. $\square$

We use Proposition 3.6 and Proposition 6.2 to further reduce to the following admissible pairs for conformal blocks on $\mathbb{P}^1$ with 3 marked points

1. $(\omega_1, \lambda_2, \lambda_3); (\omega, \lambda_2^T, \lambda_3^T)$, where $\lambda_2, \lambda_3 \in \mathcal{Y}_{r,s}$ and λ_2 is obtained by λ_3 either by adding or deleting a box.
2. $(\omega_1, \lambda, \lambda); (\omega_1, \lambda^T, \sigma(\lambda^T))$, where $\lambda \in \mathcal{Y}_{r,s}$ and $(\lambda, L_r) \neq 0$.

The rank-level duality in these cases has been proved in Section 10.2 and Section 10.3. This completes the proof of Theorem 7.1.

11. Key Lemmas

Lemma 11.1. *Let $\xi = \exp(\frac{\pi\sqrt{-1}}{2(r+s)})$. Consider the matrix W whose (i,j)-th entry is the complex number $(\xi^{i(2j-1)} - \xi^{-i(2j-1)})$.*

$$WW^T = \begin{pmatrix} c & & & \\ & \ddots & & \\ & & c & \\ & & & 2c \end{pmatrix},$$

where $c = -2(r+s)$.

Let U be a partition of $\{1, \ldots, r+s\}$ such that $r+s \in U$ and $|U| = r$. Let P be the permutation matrix associated to the permutation (U, U^c).

$$PWW^TP^{-1} = \begin{pmatrix} 2c & & & \\ & c & & \\ & & \ddots & \\ & & & c \end{pmatrix}$$

Let $A = W$ and $B = W^T$ and U,T as in Lemma 8.1. Then

$$c^s \det A_{U,T} = \operatorname{sgn}(U, U^c) \operatorname{sgn}(T, T^c) \det A \det B_{T^c,U^c}. \tag{11.1}$$

Let $[r+s]$ denote the set $\{1, 2, \ldots, r+s\}$. We define the following sets

- Given any $\lambda = (\lambda^1 \geq \lambda^2 \geq \cdots \geq \lambda^r) \in \mathcal{Y}_{r,s}$, we define $\alpha^i = \lambda^i + r + 1 - i$ and $[\alpha] = \{\alpha^1 > \alpha^2 > \cdots > \alpha^r\}$.
- Consider the complement $[\beta] = (\beta^1 > \beta^2 > \cdots > \beta^s)$ of $[\alpha]$ in $[r+s]$. We define another set $[\gamma] = (\gamma^1 > \gamma^2 > \cdots > \gamma^s)$ where $\gamma^i = ((r+s) - (\beta^{(s+1-i)} - \frac{1}{2}))$.
- Let $T = (t_1 > t_2 > \cdots > t_r)$ where $t_i = r + 1 - i$; $T' = (t'_1 > t'_2 > \cdots > t'_s)$ where $t'_i = s + 1 - i$ and $T^c = (t^c_1 > t^c_2 > \cdots > t^c_s)$ is the complement of T in $[r+s]$.
- $U = (u_1 > u_2 > \cdots > u_r)$ be a subset of $[r+s]$ of cardinality r such that $r+s \in U$ and $U^c = (u^c_1 > u^c_2 > \cdots > u^c_s)$ be the complement of U in $[r+s]$.

Then for $\lambda \in \mathcal{Y}_{r,s}$, we can write

$$\begin{aligned}\operatorname{Tr}_{V_\lambda}(\exp \pi\sqrt{-1}\frac{\mu+\rho}{r+s}) &= \frac{\det(\zeta^{u_i(\alpha^j-\frac{1}{2})}-\zeta^{-u_i(\alpha^j-\frac{1}{2})})}{\det(\zeta^{u_i(t_j-\frac{1}{2})}-\zeta^{-u_i(t_j-\frac{1}{2})})},\\ &= \frac{\det(\xi^{u_i(2\alpha^j-1)}-\xi^{-u_i(2\alpha^j-1)})}{\det(\xi^{u_i(2t_j-1)}-\xi^{-u_i(2t_j-1)})},\end{aligned}$$

where $\mu+\rho=\sum_{i=1}^r u_iL_i$.

For $\lambda^T \in \mathcal{Y}_{r,s}$, $\mu'+\rho'=\sum_{i=1}^s u_i^cL_i$ and ρ' the Weyl vector of $\mathfrak{so}(2s+1)$, we can write

$$\begin{aligned}\operatorname{Tr}_{V_{\lambda^T}}(\exp \pi\sqrt{-1}\frac{\mu'+\rho'}{r+s}) &= \frac{\det(\zeta^{u_i^c(\gamma^j)}-\zeta^{-u_i^c(\gamma^j)})}{\det(\zeta^{u_i^c(t_j'-\frac{1}{2})}-\zeta^{-u_i^c(t_j'-\frac{1}{2})})},\\ &= \frac{\det(\zeta^{u_i^c((r+s)-(\beta^j-\frac{1}{2}))}-\zeta^{-u_i^c((r+s)-(\beta^j-\frac{1}{2}))})}{\det(\zeta^{u_i^c((r+s)-(t_j^c-\frac{1}{2}))}-\zeta^{-u_i^c((r+s)-(t_j^c-\frac{1}{2}))})},\\ &= \frac{\det(\zeta^{-u_i^c(r+s)}(\zeta^{u_i^c(\beta^j-\frac{1}{2})}-\zeta^{-u_i^c(\beta^j-\frac{1}{2})}))}{\det(\zeta^{-u_i^c(r+s)}(\zeta^{u_i^c(t_j^c-\frac{1}{2})}-\zeta^{-u_i^c(t_j^c-\frac{1}{2})}))},\\ &= \frac{\det(\zeta^{u_i^c(\beta^j-\frac{1}{2})}-\zeta^{-u_i^c(\beta^j-\frac{1}{2})})}{\det(\zeta^{u_i^c(t_j^c-\frac{1}{2})}-\zeta^{-u_i^c(t_j^c-\frac{1}{2})})},\\ &= \frac{\det(\xi^{u_i^c(2\beta^j-1)}-\xi^{-u_i^c(2\beta^j-1)})}{\det(\xi^{u_i^c(2t_j^c-1)}-\xi^{-u_i^c(2t_j^c-1)})}.\end{aligned}$$

By applying 11.1, we get the following:

Lemma 11.2.

$$\operatorname{Tr}_{V_\lambda}(\exp \pi\sqrt{-1}\frac{\mu+\rho}{r+s}) = \frac{\operatorname{sgn}([\alpha],[\beta])}{\operatorname{sgn}(T,T^c)}\operatorname{Tr}_{V_{\lambda^T}}(\exp \pi\sqrt{-1}\frac{\mu'+\rho'}{r+s}).$$

The following can be checked by direct calculation

Lemma 11.3.

$$\operatorname{sgn}([\alpha],[\beta]) = (-1)^{\frac{r(r-1)}{2}+\frac{s(s-1)}{2}+|\lambda|}.$$

$$\operatorname{sgn}(T,T^c) = (-1)^{\frac{r(r-1)}{2}+\frac{s(s-1)}{2}}.$$

Thus we have the following equality:

$$\operatorname{Tr}_{V_\lambda}(\exp \pi\sqrt{-1}\frac{\mu+\rho}{r+s}) = (-1)^{|\lambda|}\operatorname{Tr}_{V_{\lambda^T}}(\exp \pi\sqrt{-1}\frac{\mu'+\rho'}{r+s}). \tag{11.2}$$

Let $\xi=\exp\frac{\pi\sqrt{-1}}{4(r+s)}$ Then the following equality holds for any integer a and b.

$$\xi^{(2(r+s)-(2a-1))(2(r+s)-(2b-1))} = (-1)^{(a+b)}\xi^{(2a-1)(2b-1)} \tag{11.3}$$

Lemma 11.4. *Let* $\xi = \exp(\frac{\pi\sqrt{-1}}{4(r+s)})$. *Consider the matrix* W *whose* (i,j)*-th entry is the complex number* $(\xi^{(2i-1)(2j-1)} - \xi^{-(2i-1)(2j-1)})$.

$$WW^T = \begin{pmatrix} c & & & \\ & \ddots & & \\ & & c & \\ & & & c \end{pmatrix},$$

where $c = -2(r+s)$.

Let U be a partition of $\{1,\dots,r+s\}$ such that $|U| = r$. Let $A = W$, $B = W^T$ and U, T as in Lemma 8.1. Then

$$c^s \det A_{U,T} = \operatorname{sgn}(U,U^c)\operatorname{sgn}(T,T^c)\det A \det B_{T^c,U^c}. \tag{11.4}$$

Let $U' = (u'_1 > u'_2 > \cdots > u'_r)$ be a subset of $[r+s]$ of cardinality r, $U'^c = (u'^c_1 > \cdots > u'^c_s)$ be the complement of U' in $[r+s]$ and $\mu + \rho = \sum_{i=1}^r (u'_i - \frac{1}{2})L_i$. Then for $\lambda \in \mathcal{Y}_{r,s}$, we can write

$$\begin{aligned} \operatorname{Tr}_{V_\lambda}(\exp \pi\sqrt{-1}\frac{\mu+\rho}{r+s}) &= \frac{\det(\zeta^{(u'_i-\frac{1}{2})(\alpha^j-\frac{1}{2})} - \zeta^{-(u'_i-\frac{1}{2})(\alpha^j-\frac{1}{2})})}{\det(\zeta^{(u'_i-\frac{1}{2})(t_j-\frac{1}{2})} - \zeta^{-(u'_i-\frac{1}{2})(t_j-\frac{1}{2})})}, \\ &= \frac{\det(\xi^{(2u'_i-1)(2\alpha^j-1)} - \xi^{-(2u'_i-1)(2\alpha^j-1)})}{\det(\xi^{(2u'_i-1)(2t_j-1)} - \xi^{-(2u'_i-1)(2t_j-1)})}. \end{aligned}$$

For $\lambda^T \in \mathcal{Y}_{r,s}$, $\mu' + \rho' = \sum_{i=1}^s ((r+s+\frac{1}{2}) - u'^c_i)L_i$ and ρ' the Weyl vector of $\mathfrak{so}(2s+1)$, we can write the following:

$$\begin{aligned} \operatorname{Tr}_{V_{\lambda^T}}(\exp \pi\sqrt{-1}\frac{\mu'+\rho'}{r+s}) &= \frac{\det(\zeta^{((r+s)-(u'^c_i-\frac{1}{2}))((r+s)-(\beta^j-\frac{1}{2}))} - \zeta^{((r+s)-(u'^c_i-\frac{1}{2}))((r+s)-(\beta^j-\frac{1}{2}))})}{\det(\zeta^{((r+s)-(u'^c_i-\frac{1}{2}))((r+s)-(t^c_j-\frac{1}{2}))} - \zeta^{((r+s)-(u'^c_i-\frac{1}{2}))((r+s)-(t^c_j-\frac{1}{2}))})}, \\ &= \frac{(-1)^{\sum_{i=1}^s (u'^c_i+\beta_i)}}{(-1)^{\sum_{i=1}^s (u'^c_i+t^c_i)}} \frac{\det(\xi^{(2u'^c_i-1)(2\beta^j-1)} - \xi^{-(2u'^c_i-1)(2\beta^j-1)})}{\det(\xi^{(2u'^c_i-1)(2t^c_j-1)} - \xi^{-(2u'^c_i-1)(2t^c_j-1)})}, \\ &= \frac{(-1)^{\sum_{i=1}^s (\beta_i)}}{(-1)^{\sum_{i=1}^s (t^c_i)}} \frac{\det(\xi^{(2u'^c_i-1)(2\beta^j-1)} - \xi^{-(2u'^c_i-1)(2\beta^j-1)})}{\det(\xi^{(2u'^c_i-1)(2t^c_j-1)} - \xi^{-(2u'^c_i-1)(2t^c_j-1)})}, \\ &= (-1)^{|\lambda|} \frac{\det(\xi^{(2u'^c_i-1)(2\beta^j-1)} - \xi^{-(2u'^c_i-1)(2\beta^j-1)})}{\det(\xi^{(2u'^c_i-1)(2t^c_j-1)} - \xi^{-(2u'^c_i-1)(2t^c_j-1)})}. \end{aligned}$$

From 11.4, we get the following:

Lemma 11.5.

$$\operatorname{Tr}_{V_\lambda}(\exp \pi\sqrt{-1}\frac{\mu+\rho}{r+s}) = \operatorname{Tr}_{V_{\lambda^T}}(\exp \pi\sqrt{-1}\frac{\mu'+\rho'}{r+s})$$

11.1. **Some Trace calculations.** Let $\zeta = \exp(\frac{\pi\sqrt{-1}}{r+s})$ and $U = (u_1 > u_2 > \cdots > u_r)$ be a subset of $[r+s]$ of cardinality r. Then we have the following

$$\begin{aligned} \zeta^{(u_i-\frac{1}{2})((2r+1)+s-\frac{1}{2})} - \zeta^{-(u_i-\frac{1}{2})((2r+1)+s-\frac{1}{2})} &= \zeta^{(u_i-\frac{1}{2})(2(r+s)-(s-\frac{1}{2}))} - \zeta^{-(u_i-\frac{1}{2})(2(r+s)-(s-\frac{1}{2}))}, \\ &= -\big(\zeta^{-(u_i-\frac{1}{2})(s-\frac{1}{2})} - \zeta^{(u_i-\frac{1}{2})(s-\frac{1}{2})}\big), \\ &= \zeta^{(u_i-\frac{1}{2})(s-\frac{1}{2})} - \zeta^{-(u_i-\frac{1}{2})(s-\frac{1}{2})}. \end{aligned}$$

The proof of the following Lemma follows from the above and the Weyl character formula.

Lemma 11.6. *Consider the dominant weight $\lambda = (2s+1)\omega_1$ of $\mathfrak{so}(2r+1)$ of level $2s+1$. Let $U = (u_1 > u_2 > \cdots > u_r)$ be a subset of $[r+s]$ or cardinality r and $\mu+\rho = \sum_{i=1}^{r}(u_i - \frac{1}{2})L_i$. Then*

$$\mathrm{Tr}_{\lambda}(\exp(\pi\sqrt{-1}\frac{\mu+\rho}{r+s})) = 1.$$

Let $\zeta = \exp(\frac{\pi\sqrt{-1}}{r+s})$ and $U = (u_1 > u_2 > \cdots > u_r)$ be a subset of $[r+s]$ of cardinality r. Then we have the following

$$\begin{aligned}
\zeta^{u_i((2r+1)+s-\frac{1}{2})} - \zeta^{-u_i((2r+1)+s-\frac{1}{2})} &= \zeta^{u_i(2(r+s)-(s-\frac{1}{2}))} - \zeta^{-u_i(2(r+s)-(s-\frac{1}{2}))}, \\
&= \zeta^{-u_i(s-\frac{1}{2})} - \zeta^{u_i(s-\frac{1}{2})}, \\
&= -\big(\zeta^{u_i(s-\frac{1}{2})} - \zeta^{-u_i(s-\frac{1}{2})}\big).
\end{aligned}$$

The proof of the following Lemma follows from the above and the Weyl character formula

Lemma 11.7. *Consider the dominant weight $\lambda = (2s+1)\omega_1$ of $\mathfrak{so}(2r+1)$ of level $2s+1$. Let $U = (u_1 > u_2 > \cdots > u_r)$ be a subset of $[r+s]$ or cardinality r and $\mu+\rho = \sum_{i=1}^{r} u_i L_i$. Then*

$$\mathrm{Tr}_{\lambda}(\exp(\pi\sqrt{-1}\frac{\mu+\rho}{r+s})) = -1.$$

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